

Boundedness of maximal operator, Hardy operator and Sobolev's inequalities on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces

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ABSTRACT. Our aim in this paper is to deal with the boundedness of the Hardy-Littlewood maximal operator and the Hardy operator on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces and to establish a generalization of Sobolev's inequalities for Riesz potentials of functions in such spaces.

1. Introduction

Let $\mathbf{R}^N$ be the Euclidean space and let $B(x, r)$ denote the open ball centered at $x \in \mathbf{R}^N$ with radius $r > 0$.

In harmonic analysis, the maximal operator is a classical tool when studying Sobolev functions and partial differential equations. This also plays a central role in the study of differentiation, singular integrals, smoothness of functions and so on (see [8, 28, 50], etc.). It is well known that the maximal operator is bounded on the Lebesgue space $L^p(\mathbf{R}^N)$ if $p > 1$ (see [50]). The boundedness of the maximal operator was studied on Morrey spaces in [11, 42], on Orlicz-Morrey spaces in [44], and also on non-homogeneous Herz spaces in [29]. For Morrey spaces, which were introduced to estimate solutions of partial differential equations, we refer to [40, 46].

One of the important applications of the boundedness of the maximal operator is Sobolev's inequality; in classical Lebesgue spaces, we know Sobolev's inequality:

$$\|I_\alpha f\|_{L^{p^*}(\mathbf{R}^N)} \leq C \|f\|_{L^p(\mathbf{R}^N)}$$

for $f \in L^p(\mathbf{R}^N)$, $0 < \alpha < N$ and $1 < p < N/\alpha$, where I_α is the Riesz kernel of order α and $1/p^* = 1/p - \alpha/N$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequality

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ity for Morrey spaces was given by D. R. Adams [1] (also [11, 42]), and then the result was extended to Orlicz-Morrey spaces in [43]. See also [29] for non-homogeneous Herz spaces and [20] for non-homogeneous central Morrey spaces. For local Morrey-type spaces, we refer the reader to [9, 10] and so on.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth condition. For a survey, see [15, 19]. The boundedness of the maximal and Riesz potential operators were studied for variable exponent Lebesgue spaces $L^{p(\cdot)}$ (see [16, 17, 18]), variable exponent Morrey spaces (see [4, 22, 23, 34, 39]), Herz spaces with variable exponents (see [3, 27, 47]), local variable exponent Morrey type spaces (see [23, 24]) and non-homogeneous central Morrey spaces of variable exponent (see [38]).

Recently, the boundedness of the maximal and Riesz potential operators were studied for Herz-Morrey spaces with variable exponents (see [35, 36]) and non-homogeneous central Herz-Morrey-Orlicz spaces in the constant exponent case (see [37]).

Let Ω be a measurable set in $\mathbf{R}^N$. Given a general function $\Phi(x, t)$ satisfying certain conditions, we consider the associated Musielak-Orlicz space (cf. [41])

$$L^\Phi(\Omega) = \left\{ f \in L^1_{loc}(\Omega); \int_{\Omega} \Phi(y, |f(y)|) dy < \infty \right\},$$

which is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0; \int_{\Omega} \bar{\Phi}(y, |f(y)|/\lambda) dy \leq 1 \right\}$$

(see Section 2 for the definitions of Φ and $\bar{\Phi}$). For the recent development of the theory of PDEs in Musielak-Orlicz spaces and Herz spaces with variable exponents, we refer to [7, 12, 25, 48] and so on. Let $\omega(r) : (0, \infty) \rightarrow (0, \infty)$ be almost monotone on $(0, \infty)$ satisfying the doubling condition. Let $0 < q < \infty$. Given $\Phi(x, t)$ and $\omega(r)$, we denote by $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ the class of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0, 2))} + \left(\int_1^\infty (\omega(r) \|f\|_{L^\Phi(A(0, r))})^q \frac{dr}{r} \right)^{1/q} < \infty,$$

where $A(0, r) = B(0, 2r) \setminus B(0, r)$. The space $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ is referred to as a non-homogeneous central Herz-Morrey-Musielak-Orlicz space (see Section 2).

Our first aim in this paper is to study the boundedness of the maximal operator on non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces

$\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ (Theorem 1 below), as an extension of [36, 37]. To this end, we apply the boundedness of the Hardy-Littlewood maximal operator on L^Φ given in [30]. The case when $q = \infty$ was treated in [45], as an extension of [35].

Next we study the boundedness of the Hardy operators $\hat{H}_\beta^\infty$ and $\hat{H}_\beta^0$ on $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ (Theorems 2 and 3 below). See Section 4 for the definitions of $\hat{H}_\beta^\infty$ and $\hat{H}_\beta^0$.

As an application of the boundedness of the maximal operator, we establish Sobolev's inequality for Riesz potentials $I_\alpha f$ of functions in $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ (Theorem 4 below), as an extension of [36, 37]. When $q = \infty$, we refer to [35, 45].

Further, we discuss Sobolev's inequality for generalized Riesz potentials $I_{\alpha, k} f$ of functions in $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ (Theorem 5 below), as an extension of [35, 36, 37]. See Section 6 for the definition of $I_{\alpha, k} f$.

In Section 7, in connection with the study in [21, 24], we investigate the space $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ and its complementary space $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$.

In Section 8, we treat the case q is variable.

Throughout this paper, let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \leq g \leq Ch$ for some constant $C > 0$.

2. Preliminaries

We consider a function

$$\Phi(x, t) = t\phi(x, t) : \mathbf{R}^N \times [0, \infty) \rightarrow [0, \infty)$$

satisfying the following conditions $(\Phi 1)$ – $(\Phi 4)$:

- $(\Phi 1)$ $\phi(\cdot, t)$ is measurable on $\mathbf{R}^N$ for each $t \geq 0$ and $\phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in \mathbf{R}^N$;
- $(\Phi 2)$ there exists a constant $A_1 \geq 1$ such that

$$A_1^{-1} \leq \phi(x, 1) \leq A_1 \quad \text{for all } x \in \mathbf{R}^N;$$

- $(\Phi 3)$ $\phi(x, \cdot)$ is uniformly almost increasing; namely there exists a constant $A_2 \geq 1$ such that

$$\phi(x, t) \leq A_2 \phi(x, s) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 \leq t < s;$$

- $(\Phi 4)$ there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$ imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x, t) \leq \sup_{x \in \mathbf{R}^N} \phi(x, t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbf{R}^N$, then $(\Phi 3)$ holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\bar{\Phi}(x, t) = \int_0^t \bar{\phi}(x, r) dr$$

for $x \in \mathbf{R}^N$ and $t \geq 0$. Then $\bar{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2A_3} \Phi(x, t) \leq \bar{\Phi}(x, t) \leq A_2 \Phi(x, t) \quad (1)$$

for all $x \in \mathbf{R}^N$ and $t \geq 0$.

By $(\Phi 3)$, we see that

$$\Phi(x, at) \begin{cases} \leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1, \\ \geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1. \end{cases}$$

We shall also consider the following conditions for $\Phi(x, t)$: Let $p \geq 1$, $q \geq 1$, $\eta > 0$ and $\tau > 0$.

$(\Phi 3; 0; p)$ $t \mapsto t^{-p} \Phi(x, t)$ is uniformly almost increasing on $(0, 1]$, namely there exists a constant $A_{2,0,p} \geq 1$ such that

$$t_1^{-p} \Phi(x, t_1) \leq A_{2,0,p} t_2^{-p} \Phi(x, t_2) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t_1 < t_2 \leq 1;$$

$(\Phi 3; \infty; q)$ $t \mapsto t^{-q} \Phi(x, t)$ is uniformly almost increasing on $[1, \infty)$, namely there exists a constant $A_{2,\infty,q} \geq 1$ such that

$$t_1^{-q} \Phi(x, t_1) \leq A_{2,\infty,q} t_2^{-q} \Phi(x, t_2) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } 1 \leq t_1 < t_2;$$

$(\Phi 5; \eta)$ for every $\gamma > 0$, there exists a constant $B_{\gamma,\eta} \geq 1$ such that

$$\Phi(x, t) \leq B_{\gamma,\eta} \Phi(y, t)$$

whenever $x, y \in \mathbf{R}^N$, $|x - y| \leq \gamma t^{-\eta}$ and $t \geq 1$;
 $(\Phi 6; \tau)$ there exist a function g on $\mathbf{R}^N$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) \leq 1$ for all $x \in \mathbf{R}^N$, $g \in L^\tau(\mathbf{R}^N)$ and

$$B_\infty^{-1} \Phi(x, t) \leq \Phi(x', t) \leq B_\infty \Phi(x, t)$$

whenever $x, x' \in \mathbf{R}^N$, $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

Note that $(\Phi 3; 0; 1) + (\Phi 3; \infty; 1) = (\Phi 3)$. If $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, then it satisfies $(\Phi 3; 0; p')$ for $1 \leq p' \leq p$; if $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$, then it satisfies $(\Phi 3; \infty; q')$ for $1 \leq q' \leq q$.

If $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, then

$$\Phi(x, t) \leq A_1 A_{2,0,p} t^p \quad \text{for } 0 \leq t \leq 1;$$

if $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$, then

$$\Phi(x, t) \geq (A_1 A_{2,\infty,q})^{-1} t^q \quad \text{for } t \geq 1.$$

If $\Phi(x, t)$ satisfies $(\Phi 5; \eta)$, then it satisfies $(\Phi 5; \eta')$ for all $\eta' \geq \eta$; if $\Phi(x, t)$ satisfies $(\Phi 6; \tau)$, then it satisfies $(\Phi 6; \tau')$ for all $\tau' \geq \tau$.

In the following examples, we use the notation

$$f^- := \inf_{x \in \mathbf{R}^N} f(x) \quad \text{and} \quad f^+ := \sup_{x \in \mathbf{R}^N} f(x)$$

for a measurable function f on $\mathbf{R}^N$.

EXAMPLE 1. Let $p_i(\cdot)$, $i = 1, 2$ and $q_{i,j}(\cdot)$, $j = 1, \dots, k_i$, be real valued measurable functions on $\mathbf{R}^N$ such that $p_i^- > 1$ and $q_{i,j}^- > -\infty$, $i = 1, 2$, $j = 1, \dots, k_i$.

Set $L_c(t) = \log(c + t)$ for $c > 1$ and $t \geq 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$. Let

$$\Phi(x, t) = \begin{cases} t^{p_1(x)} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/t))^{-q_{1,j}(x)} & \text{if } 0 \leq t \leq 1, \\ t^{p_2(x)} \prod_{j=1}^{k_2} (L_{e-1}^{(j)}(t))^{q_{2,j}(x)} & \text{if } t \geq 1. \end{cases}$$

Then, $\Phi(x, t)$ satisfies $(\Phi 1)$, $(\Phi 2)$ and $(\Phi 3)$. It satisfies $(\Phi 3; 0; p)$ for $1 \leq p < p_1^-$ in general and for $1 \leq p \leq p_1^-$ in case $q_{1,j}^- \geq 0$ for all $j = 1, \dots, k_1$; it satisfies $(\Phi 3; \infty; q)$ for $1 \leq q < p_2^-$ in general and for $1 \leq q \leq p_2^-$ in case $q_{2,j}^- \geq 0$ for all $j = 1, \dots, k_2$.

Moreover, we see that $\Phi(x, t)$ satisfies $(\Phi 5; \eta)$ for every $\eta > 0$ if $p_2(\cdot)$ is log-Hölder continuous, namely

$$|p_2(x) - p_2(y)| \leq \frac{C_p}{L_e(1/|x - y|)} \quad (x, y \in \mathbf{R}^N)$$

with a constant $C_p \geq 0$ and $q_{2,j}(\cdot)$ is $(j+1)$ -log-Hölder continuous, namely

$$|q_{2,j}(x) - q_{2,j}(y)| \leq \frac{C_j}{L_e^{(j+1)}(1/|x - y|)} \quad (x, y \in \mathbf{R}^N)$$

with constants $C_j \geq 0$, for each $j = 1, \dots, k_2$.

Finally, we see that $\Phi(x, t)$ satisfies $(\Phi 6; \tau)$ for every $\tau > 0$ with $g(x) = (1 + |x|)^{-(N+1)/\tau}$ if $p_1(\cdot)$ is log-Hölder continuous at ∞ , namely

$$|p_1(x) - p_1(x')| \leq \frac{C_{p, \infty}}{L_e(|x|)}$$

whenever $|x'| \geq |x|$ ($x, x' \in \mathbf{R}^N$) with a constant $C_{p, \infty} \geq 0$, and $q_{1,j}(\cdot)$ is $(j+1)$ -log-Hölder continuous at ∞ , namely

$$|q_{1,j}(x) - q_{1,j}(x')| \leq \frac{C'_j}{L_e^{(j+1)}(|x|)}$$

whenever $|x'| \geq |x|$ ($x, x' \in \mathbf{R}^N$) with a constant $C'_j \geq 0$, for each $j = 1, \dots, k_1$. In fact, if $(1 + |x|)^{-(N+1)/\tau} < t \leq 1$, then $t^{-|p_1(x) - p_1(x')|} \leq e^{(N+1)C_{p, \infty}/\tau}$ for $|x'| \geq |x|$ and $(L_{e-1}^{(j)}(1/t))^{|q_{1,j}(x) - q_{1,j}(x')|} \leq C(N, C'_j)$ for $|x'| \geq |x|$.

The following example shows that our conditions are satisfied by the double phase functional with variable exponents.

EXAMPLE 2. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [5, 6, 7, 13, 14] have studied the double phase functional

$$\Phi(x, t) = t^p + a(x)t^q,$$

where $1 < p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0, 1]$. In [31], we studied the double phase functional with variable exponents:

$$\Phi(x, t) = t^{p(x)} + a(x)t^{q(x)}, \quad x \in \mathbf{R}^N, t \geq 0,$$

where $p(\cdot)$ and $q(\cdot)$ are real valued functions on $\mathbf{R}^N$ such that $p(x) < q(x)$ for $x \in \mathbf{R}^N$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0, 1]$. This $\Phi(x, t)$ satisfies $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3)$, $(\Phi 3; 0; p^-)$, $(\Phi 3; \infty; p^-)$ and $(\Phi 5; \eta)$ for $\eta \geq \sup_{\{x: a(x) > 0\}} (q(x) - p(x))/\theta$ if $1 \leq p^- \leq p^+ < \infty$, $1 \leq q^- \leq q^+ < \infty$, $p(\cdot)$ and $q(\cdot)$ are log-Hölder continuous. Further it satisfies $(\Phi 6; \tau)$ with $g(x) = (1 + |x|)^{-(N+1)/\tau}$ for every $\tau > 0$ if $p(\cdot)$ is log-Hölder continuous at ∞ . See [31] for details.

Let Ω be a measurable set in $\mathbf{R}^N$. From now on, we assume that $\Phi(x, t)$ satisfies $(\Phi 1)$, $(\Phi 2)$, $(\Phi 3)$ and $(\Phi 4)$. Then the associated Musielak-Orlicz space

$$L^\Phi(\Omega) = \left\{ f \in L^1_{loc}(\Omega); \int_{\Omega} \Phi(y, |f(y)|) dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \lambda > 0; \int_{\Omega} \bar{\Phi}(y, |f(y)|/\lambda) dy \leq 1 \right\}$$

(cf. [41]).

REMARK 1. The Musielak-Orlicz spaces $L^\Phi(\Omega)$ include the following spaces:

- Orlicz spaces defined by Young functions satisfying the doubling condition;
- variable exponent Lebesgue spaces.

REMARK 2. The dominated convergence theorem and $(\Phi 4)$ yield

$$\int_{\Omega} \bar{\Phi} \left(y, \frac{|f(y)|}{\|f\|_{L^\Phi(\Omega)}} \right) dy = 1.$$

We consider a function $\omega(r) : (0, \infty) \rightarrow (0, \infty)$ satisfying the following conditions $(\omega 1)$ and $(\omega 2)$:

- $(\omega 1)$ $\omega(\cdot)$ is almost monotone on $(0, \infty)$; that is, $\omega(\cdot)$ is almost increasing on $(0, \infty)$ or $\omega(\cdot)$ is almost decreasing on $(0, \infty)$; namely there exists a constant $c_1 > 0$ such that

$$\omega(r) \leq c_1 \omega(s) \quad \text{for all } 0 < r < s$$

or

$$\omega(s) \leq c_1 \omega(r) \quad \text{for all } 0 < r < s$$

respectively;

- $(\omega 2)$ $\omega(\cdot)$ is doubling on $(0, \infty)$; that is, there exists a constant $c_2 > 1$ such that

$$c_2^{-1} \omega(r) \leq \omega(2r) \leq c_2 \omega(r) \quad \text{for all } r > 0.$$

Let $0 < q < \infty$. Given $\Phi(x, t)$ and $\omega(r)$ as above, we denote by $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ the class of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0, 2))} + \left(\int_1^\infty (\omega(r) \|f\|_{L^\Phi(A(0, r))})^q \frac{dr}{r} \right)^{1/q} < \infty,$$

where $A(0, r) = B(0, 2r) \setminus B(0, r)$. The space $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ is referred to as a non-homogeneous central Herz-Morrey-Musielak-Orlicz space.

REMARK 3. The non-homogeneous central Herz-Morrey-Musielak-Orlicz spaces $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ include the following spaces:

- non-homogeneous Herz spaces introduced in [26];
- local Morrey-type spaces introduced in [9];
- non-homogeneous central Herz-Morrey-Orlicz spaces introduced in [37] where $\Phi(x, t) = \Phi(t)$;
- non-homogeneous central Herz-Morrey spaces with variable exponents introduced in [36] where $\Phi(x, t) = t^{p(x)}$.

LEMMA 1. *For $1/2 < a < 1 < b < 2$ with $2a \geq b$, there exists a constant $C > 0$ such that*

$$\int_{at}^{bt} (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} \geq C (\omega(t) \|f\|_{L^{\Phi(A(0,t))}})^q \quad (2)$$

for all $t > 0$.

PROOF. For $1/2 < a < 1 < b < 2$ with $2a \geq b$, we have

$$\int_{at}^t (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} \geq C (\omega(t) \|f\|_{L^{\Phi(B(0,2at) \setminus B(0,t))}})^q$$

and

$$\int_t^{bt} (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} \geq C (\omega(t) \|f\|_{L^{\Phi(B(0,2t) \setminus B(0,bt))}})^q,$$

so that we obtain

$$\begin{aligned} & (\omega(t) \|f\|_{L^{\Phi(A(0,t))}})^q \\ & \leq (\omega(t) \|f\|_{L^{\Phi(B(0,2t) \setminus B(0,bt))}} + \omega(t) \|f\|_{L^{\Phi(B(0,2at) \setminus B(0,t))}})^q \\ & \leq C \left\{ \int_t^{bt} (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} + \int_{at}^t (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} \right\} \\ & = C \int_{at}^{bt} (\omega(r) \|f\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r}. \quad \square \end{aligned}$$

LEMMA 2. *For a bounded measurable set Ω , there exist constants C_Ω and C'_Ω such that*

$$\int_\Omega |f(x)| dx \leq C_\Omega \|f\|_{L^\Phi(\Omega)} \leq C'_\Omega \|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \quad (3)$$

for all $f \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

PROOF. If $\|f\|_{L^\Phi(\Omega)} \leq 1$, then

$$\int_\Omega |f(x)| dx \leq |\Omega| + 2A_1 A_3 \int_\Omega \bar{\Phi}(x, |f(x)|) dx \leq |\Omega| + 2A_1 A_3$$

by $(\Phi 2)$, convexity of $\bar{\Phi}(x, \cdot)$ and (1), where $|\Omega|$ denotes the Lebesgue measure of Ω . This shows the first inequality in (3).

Next, suppose $f \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ and $\Omega \subset B(0, 2^k)$ ($k \geq 1$). Then

$$\begin{aligned} \|f\|_{L^\Phi(\Omega)} &\leq \|f\|_{L^\Phi(B(0, 2))} + \sum_{j=1}^{k-1} \|f\|_{L^\Phi(A(0, 2^j))} \\ &\leq \|f\|_{L^\Phi(B(0, 2))} + C_k \sum_{j=1}^{k-1} \omega(2^j) \|f\|_{L^\Phi(A(0, 2^j))}, \end{aligned}$$

where $C_k^{-1} = \inf_{2 \leq r \leq 2^{k-1}} \omega(r) > 0$. Then, using Lemma 1, we obtain the second inequality in (3). $\square$

LEMMA 3 (cf. [30, Lemma 5.1]). *Let $F(x, t)$ be a positive function on $\mathbf{R}^N \times (0, \infty)$ satisfying the following conditions:*

(F1) *$F(x, \cdot)$ is strictly increasing and continuous on $(0, \infty)$ for each $x \in \mathbf{R}^N$;*

(F2) *there exists a constant $K_1 \geq 1$ such that*

$$K_1^{-1} \leq F(x, 1) \leq K_1 \quad \text{for all } x \in \mathbf{R}^N;$$

(F3) *$t \mapsto t^{-\varepsilon} F(x, t)$ is uniformly almost increasing for $\varepsilon > 0$; namely there exists a constant $K_2 \geq 1$ such that*

$$t^{-\varepsilon} F(x, t) \leq K_2 s^{-\varepsilon} F(x, s) \quad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t < s;$$

(F4) *there exists a constant $K_3 > 1$ such that*

$$F(x, 2t) \leq K_3 F(x, t) \quad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Let $F^{-1}(x, \cdot)$ be the inverse function of $F(x, \cdot)$. Then:

(1) *$F^{-1}(x, \cdot)$ is strictly increasing.*

(2)

$$F^{-1}(x, \lambda t) \leq (K_2 \lambda)^{1/\varepsilon} F^{-1}(x, t)$$

for all $x \in \mathbf{R}^N$, $t > 0$ and $\lambda \geq 1$.

(3)

$$F^{-1}(x, \lambda t) \leq 2\lambda^{1/\log_2 K_3} F^{-1}(x, t)$$

for all $x \in \mathbf{R}^N$, $t > 0$ and $0 < \lambda < 1$.

(4)

$$\min \left\{ 1, \left(\frac{t}{K_1 K_2} \right)^{1/\varepsilon} \right\} \leq F^{-1}(x, t) \leq \max \{ 1, (K_1 K_2 t)^{1/\varepsilon} \}$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

REMARK 4. $F(x, t) = \bar{\Phi}(x, t)$ satisfies (F1), (F2), (F3) and (F4) with $K_1 = A_1 \max\{A_2, 2A_3\}$, $K_2 = 1$, $K_3 = 2A_3$ and $\varepsilon = 1$.

We also consider a convex function $\Phi_\infty(t) = t\phi_\infty(t) : [0, \infty) \rightarrow [0, \infty)$ such that $\phi_\infty(t) > 0$ for $t > 0$, $\phi_\infty(t)$ is increasing on $[0, \infty)$ and satisfies the doubling condition and

($\Phi_\infty 1$) there exists a constant $Q \geq 1$ such that

$$Q^{-1}\Phi(x, t) \leq \Phi_\infty(t) \leq Q\Phi(x, t) \quad \text{whenever } g(x) \leq t \leq 1$$

for g in condition ($\Phi 6; \tau$).

REMARK 5. Note from ($\Phi_\infty 1$) that for $c_1, c_2 > 0$, there exists a constant $Q \geq 1$ such that

$$Q^{-1}\Phi(x, t) \leq \Phi_\infty(t) \leq Q\Phi(x, t) \quad \text{whenever } c_1g(x) \leq t \leq c_2$$

for g in condition ($\Phi 6; \tau$).

REMARK 6. Suppose $\Phi(x, t)$ satisfies ($\Phi 6; \tau$). Set

$$\Phi_\infty(t) = \limsup_{|x| \rightarrow \infty} \bar{\Phi}(x, t) \quad \text{and} \quad \phi_\infty(t) = \Phi_\infty(t)/t.$$

Then note that $\phi_\infty(t) > 0$ for $t > 0$, $\phi_\infty(t)$ is increasing on $[0, \infty)$ and satisfies the doubling condition. Further, by ($\Phi 6; \tau$), we find that $\Phi_\infty(t)$ satisfies ($\Phi_\infty 1$).

We denote by χ_E the characteristic function of E and by $\Phi_\infty^{-1}(t)$ the inverse of $\Phi_\infty(t)$.

LEMMA 4. Assume

($\Phi_\infty 2$) there exists a constant $Q \geq 1$ such that

$$\Phi_\infty(g(x)) \leq Q(1 + |x|)^{-N}$$

for all $x \in \mathbf{R}^N$.

Then there is a constant $C > 0$ such that

$$\|\chi_{B(0, r)}\|_{L^\Phi(\mathbf{R}^N)} \leq C\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$$

for all $r \geq 1$.

PROOF. Note from ($\Phi_\infty 2$) and Lemma 3 (2) that

$$g(x) \leq C\Phi_\infty^{-1}((1 + |x|)^{-N}) \leq C\Phi_\infty^{-1}(1)$$

for all $x \in \mathbf{R}^N$.

Let $R \geq 1/2$. We have by $(\Phi 3)$, $(\Phi 4)$, Lemma 3 (2) and $(\Phi_\infty 1)$

$$\begin{aligned} \int_{A(0,R)} \Phi(y, \Phi_\infty^{-1}(R^{-N})) dy &\leq C \int_{A(0,R)} \Phi(y, \Phi_\infty^{-1}((1+|y|)^{-N})) dy \\ &\leq C \int_{A(0,R)} \Phi_\infty(\Phi_\infty^{-1}((1+|y|)^{-N})) dy \\ &\leq C \int_{A(0,R)} (1+|y|)^{-N} dy \\ &\leq C. \end{aligned}$$

Hence we obtain

$$\|\chi_{A(0,R)}\|_{L^\Phi(\mathbf{R}^N)} \leq C\{\Phi_\infty^{-1}(R^{-N})\}^{-1}$$

for all $R \geq 1/2$.

Here note from Lemma 3 (4) that

$$\Phi_\infty^{-1}(R^{-N}) \leq \max\{1, CR^{-N}\} \leq C,$$

so that

$$\max\{\{\Phi_\infty^{-1}(R^{-N})\}^{-1}, 1\} \leq C\{\Phi_\infty^{-1}(R^{-N})\}^{-1}$$

for all $R \geq 1/2$.

Fix $r \geq 1$. Let j_0 be the largest integer such that $2^{-j_0+1}r \geq 1$. Now we see from Lemma 3 (3) that $t \mapsto t^{-\varepsilon}\{\Phi_\infty^{-1}(t^{-N})\}^{-1}$ is almost increasing on $(0, \infty)$ for some constant $\varepsilon > 0$, so that

$$\begin{aligned} \|\chi_{B(0,r)}\|_{L^\Phi(\mathbf{R}^N)} &\leq \sum_{j=1}^{j_0} \|\chi_{A(0,2^{-j}r)}\|_{L^\Phi(\mathbf{R}^N)} + \|\chi_{B(0,1)}\|_{L^\Phi(\mathbf{R}^N)} \\ &\leq C \left\{ \sum_{j=1}^{j_0} \{\Phi_\infty^{-1}((2^{-j}r)^{-N})\}^{-1} + 1 \right\} \\ &\leq C \left\{ r^{-\varepsilon} \{\Phi_\infty^{-1}(r^{-N})\}^{-1} \sum_{j=1}^{j_0} (2^{-j}r)^\varepsilon + 1 \right\} \\ &\leq C \{\Phi_\infty^{-1}(r^{-N})\}^{-1}, \end{aligned}$$

as required. □

REMARK 7. If $g(x) \leq C(1+|x|)^{-N}$, then $(\Phi_\infty 2)$ holds by convexity of Φ_∞ .

REMARK 8. Let $\Phi(x, t)$ and $g(x)$ be as in Example 1. Then there exist constants $p_1(\infty) > 1$ and $q_{1,j}(\infty) \in \mathbf{R}$ for $j = 1, \dots, k_1$ such that

$$\lim_{|x| \rightarrow \infty} p_1(x) = p_1(\infty) \quad \text{and} \quad \lim_{|x| \rightarrow \infty} q_{1,j}(x) = q_{1,j}(\infty).$$

Set $\bar{\phi}_\infty(t) = \sup_{0 \leq s \leq t} \{s^{p_1(\infty)-1} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/s))^{-q_{1,j}(\infty)}\}$ and

$$\Phi_\infty(t) = \int_0^t \bar{\phi}_\infty(r) \frac{dr}{r}.$$

Then $\Phi_\infty(t)$ satisfies $(\Phi_\infty 1)$ and $(\Phi_\infty 2)$ for $0 < \tau < (N+1)p_1(\infty)/N$.

LEMMA 5. Suppose that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$. Then there is a constant $C > 0$ such that

$$\frac{1}{|A(0, r)|} \int_{A(0, r)} |f(y)| dy \leq C \Phi_\infty^{-1}(r^{-N}) \|f\|_{L^\Phi(A(0, r))}$$

when $r \geq 1$ and $\|f\|_{L^\Phi(A(0, r))} < \infty$.

PROOF. Fix $r \geq 1$. Let f be a nonnegative measurable function on $A(0, r)$ satisfying $\|f\|_{L^\Phi(A(0, r))} \leq 1$. Then we have by $(\Phi 3)$

$$\begin{aligned} & \frac{1}{|A(0, r)|} \int_{A(0, r)} f(y) dy \\ & \leq \Phi_\infty^{-1}(r^{-N}) + \frac{A_2}{|A(0, r)|} \int_{A(0, r)} f(y) \frac{\varphi(y, f(y))}{\varphi(y, \Phi_\infty^{-1}(r^{-N}))} dy \\ & = \Phi_\infty^{-1}(r^{-N}) + \frac{A_2 \Phi_\infty^{-1}(r^{-N})}{|A(0, r)|} \int_{A(0, r)} \Phi(y, f(y)) \{\Phi(y, \Phi_\infty^{-1}(r^{-N}))\}^{-1} dy. \end{aligned}$$

Since

$$g(y) \leq C \Phi_\infty^{-1}((1 + |y|)^{-N}) \leq C \Phi_\infty^{-1}(r^{-N}) \leq C \Phi_\infty^{-1}(1)$$

for all $y \in A(0, r)$ by $(\Phi_\infty 2)$, we have by $(\Phi_\infty 1)$

$$\Phi(y, \Phi_\infty^{-1}(r^{-N})) \geq C r^{-N}$$

for all $y \in A(0, r)$. Hence we obtain

$$\begin{aligned} \frac{1}{|A(0, r)|} \int_{A(0, r)} f(y) dy & \leq \Phi_\infty^{-1}(r^{-N}) + C \Phi_\infty^{-1}(r^{-N}) \int_{A(0, r)} \Phi(y, f(y)) dy \\ & \leq C \Phi_\infty^{-1}(r^{-N}), \end{aligned}$$

as required. □

3. Boundedness of the maximal operator

For a locally integrable function f on $\mathbf{R}^N$, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

The mapping $f \mapsto Mf$ is called the maximal operator.

By [31, Theorem 3.1], we have the following result.

LEMMA 6. *Suppose that $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Then the maximal operator M is bounded from $L^\Phi(\mathbf{R}^N)$ into itself; namely, there is a constant $C > 0$ such that*

$$\|Mf\|_{L^\Phi(\mathbf{R}^N)} \leq C \|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

For a nonnegative function $f \in L^1_{loc}(\mathbf{R}^N)$ and a real number β , set

$$H^\beta_\beta f(r) = r^\beta \int_{\mathbf{R}^N \setminus B(0, 2r)} |y|^{-N-\beta} f(y) dy.$$

LEMMA 7. *For a real number β , suppose that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; \beta)$ $t^{\varepsilon_1 - \beta} \omega(t)^{-1} \Phi_\infty^{-1}(t^{-N})$ is almost decreasing in $[1, \infty)$ for some $\varepsilon_1 > 0$. If $0 < \varepsilon < \varepsilon_1$, then there exists a constant $C > 0$ such that*

$$H^\beta_\beta f(r) \leq Cr^\varepsilon \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$.

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \geq 1$ and $0 < \varepsilon < \varepsilon_1$.

First we consider the case $1 < q < \infty$. Then we have by Lemma 5 and Hölder's inequality

$$\begin{aligned} H^\beta_\beta f(r) &= r^\beta \sum_{j=1}^{\infty} \int_{A(0, 2^j r)} |y|^{-N-\beta} f(y) dy \\ &\leq Cr^\beta \sum_{j=1}^{\infty} (2^j r)^{-\beta} \frac{1}{|A(0, 2^j r)|} \int_{A(0, 2^j r)} f(y) dy \\ &\leq Cr^\beta \sum_{j=1}^{\infty} (2^j r)^{-\beta} \Phi_\infty^{-1}((2^j r)^{-N}) \|f\|_{L^\Phi(A(0, 2^j r))} \end{aligned}$$

$$\begin{aligned} &\leq Cr^\beta \left(\sum_{j=1}^{\infty} ((2^j r)^{\varepsilon-\beta} \omega(2^j r)^{-1} \Phi_\infty^{-1}((2^j r)^{-N}))^{q'} \right)^{1/q'} \\ &\quad \times \left(\sum_{j=1}^{\infty} ((2^j r)^{-\varepsilon} \omega(2^j r) \|f\|_{L^\Phi(A(0, 2^j r))})^q \right)^{1/q}. \end{aligned}$$

Here note from $(\Phi_\infty \omega 1; \beta)$ that

$$\begin{aligned} &\left(\sum_{j=1}^{\infty} ((2^j r)^{\varepsilon-\beta} \omega(2^j r)^{-1} \Phi_\infty^{-1}((2^j r)^{-N}))^{q'} \right)^{1/q'} \\ &\leq Cr^{\varepsilon_1-\beta} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\sum_{j=1}^{\infty} (2^j r)^{(\varepsilon-\varepsilon_1)q'} \right)^{1/q'} \\ &\leq Cr^{\varepsilon-\beta} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}). \end{aligned}$$

By (2), we have

$$\begin{aligned} &\left(\sum_{j=1}^{\infty} ((2^j r)^{-\varepsilon} \omega(2^j r) \|f\|_{L^\Phi(A(0, 2^j r))})^q \right)^{1/q} \\ &\leq C \left(\sum_{j=1}^{\infty} (2^j r)^{-\varepsilon} \int_{(2/3)2^j r}^{(4/3)2^j r} (\omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q} \\ &\leq C \left(\sum_{j=1}^{\infty} \int_{2^{j-1} r}^{2^{j+1} r} (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q} \\ &\leq C \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

Hence

$$H_\beta^\infty f(r) \leq Cr^\varepsilon \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}.$$

For the case $0 < q \leq 1$, by the fact that $(a+b)^q \leq a^q + b^q$ for all $a, b \geq 0$ instead of Hölder's inequality, we obtain the required inequality. $\square$

For a nonnegative function $f \in L_{loc}^1(\mathbf{R}^N)$ and a real number β , set

$$H_\beta^0 f(r) = r^\beta \int_{B(0, r) \setminus B(0, 1)} |y|^{-N-\beta} f(y) dy.$$

LEMMA 8. For a real number β , suppose that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; \beta)$ $t^{-\varepsilon_2 - \beta} \omega(t)^{-1} \Phi_\infty^{-1}(t^{-N})$ is almost increasing in $[1, \infty)$ for some $\varepsilon_2 > 0$.

If $0 < \varepsilon < \varepsilon_2$, then there exists a constant $C > 0$ such that

$$H_\beta^0 f(r) \leq Cr^{-\varepsilon} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_{1/2}^r (t^\varepsilon \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for all $r \geq 1$ and nonnegative functions $f \in L_{loc}^1(\mathbf{R}^N)$.

PROOF. We show only the case $1 < q < \infty$ since the remaining case is easily treated. Let $f \in L_{loc}^1(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \geq 1$ and $0 < \varepsilon < \varepsilon_2$. Let j_0 be the largest integer such that $2^{-j_0+1}r \geq 1$. We have by Lemma 5, Hölder's inequality, $(\Phi_\infty \omega 2; \beta)$ and (2)

$$\begin{aligned} H_\beta^0 f(r) &= r^\beta \sum_{j=1}^{j_0} \int_{A(0, 2^{-j}r) \setminus B(0, 1)} |y|^{-N-\beta} f(y) dy \\ &\leq Cr^\beta \sum_{j=1}^{j_0} (2^{-j}r)^{-\beta} \frac{1}{|A(0, 2^{-j}r)|} \int_{A(0, 2^{-j}r) \setminus B(0, 1)} f(y) dy \\ &\leq Cr^\beta \sum_{j=1}^{j_0} (2^{-j}r)^{-\beta} \Phi_\infty^{-1}((2^{-j}r)^{-N}) \|f \chi_{\mathbf{R}^N \setminus B(0, 1)}\|_{L^\Phi(A(0, 2^{-j}r))} \\ &\leq Cr^\beta \left(\sum_{j=1}^{j_0} ((2^{-j}r)^{-\varepsilon - \beta} \omega(2^{-j}r)^{-1} \Phi_\infty^{-1}((2^{-j}r)^{-N}))^{q'} \right)^{1/q'} \\ &\quad \times \left(\sum_{j=1}^{j_0} ((2^{-j}r)^\varepsilon \omega(2^{-j}r) \|f \chi_{\mathbf{R}^N \setminus B(0, 1)}\|_{L^\Phi(A(0, 2^{-j}r))})^q \right)^{1/q} \\ &\leq Cr^{-\varepsilon_2} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\sum_{j=1}^{j_0} (2^{-j}r)^{(\varepsilon_2 - \varepsilon)q'} \right)^{1/q'} \\ &\quad \times \left(\sum_{j=1}^{j_0} ((2^{-j}r)^\varepsilon \omega(2^{-j}r) \|f \chi_{\mathbf{R}^N \setminus B(0, 1)}\|_{L^\Phi(A(0, 2^{-j}r))})^q \right)^{1/q} \\ &\leq Cr^{-\varepsilon} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \\ &\quad \times \left(\sum_{j=1}^{j_0} ((2^{-j}r)^\varepsilon \omega(2^{-j}r) \|f \chi_{\mathbf{R}^N \setminus B(0, 1)}\|_{L^\Phi(A(0, 2^{-j}r))})^q \right)^{1/q} \end{aligned}$$

$$\begin{aligned}
&\leq Cr^{-\varepsilon}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N})\left(\int_{1/4}^r(t^{\varepsilon}\omega(t)\|f\chi_{\mathbf{R}^N\setminus B(0,1)}\|_{L^{\Phi(A(0,t))}})\frac{dt}{t}\right)^{1/q} \\
&\leq Cr^{-\varepsilon}\omega(r)^{-1}\Phi_{\infty}^{-1}(r^{-N})\left(\int_{1/2}^r(t^{\varepsilon}\omega(t)\|f\|_{L^{\Phi(A(0,t))}})^q\frac{dt}{t}\right)^{1/q},
\end{aligned}$$

which gives the required result. $\square$

We present the boundedness of the maximal operator in $\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

THEOREM 1. *Suppose that $\Phi(x,t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Phi_{\infty}(t)$ satisfies $(\Phi_{\infty} 2)$, $(\Phi_{\infty} \omega 1; 0)$ and $(\Phi_{\infty} \omega 2; -N)$. Then the maximal operator M is bounded from $\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ to itself, that is,*

$$\|Mf\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \quad \text{for all } f \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N).$$

REMARK 9. Let $\Phi(x,t)$ be as in Example 1 and let $\Phi_{\infty}(t)$ be as in Remark 8. If $\omega(r) = r^v$, then $(\Phi_{\infty} \omega 1; 0)$ and $(\Phi_{\infty} \omega 2; -N)$ hold when

$$-N/p_1(\infty) < v < N(1 - 1/p_1(\infty)).$$

PROOF (Proof of Theorem 1). Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \leq 1$. First we show

$$\int_2^{\infty} (\omega(r)\|Mf\|_{L^{\Phi(A(0,r))}})^q \frac{dr}{r} \leq C. \quad (4)$$

For $r \geq 2$, set

$$\begin{aligned}
f &= f\chi_{B(0,1)} + f\chi_{B(0,r/2)\setminus B(0,1)} + f\chi_{B(0,4r)\setminus B(0,r/2)} + f\chi_{\mathbf{R}^N\setminus B(0,4r)} \\
&= f_0 + f_{1,r} + f_{2,r} + f_{3,r}.
\end{aligned}$$

For f_0 , by Lemma 2 we have

$$Mf_0(x) \leq C|x|^{-N} \int_{B(0,1)} f(y)dy \leq C|x|^{-N}$$

for $x \in \mathbf{R}^N \setminus B(0,r)$. By Lemmas 6 and 4

$$\begin{aligned}
r^N \|\cdot\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))} &\leq C \|M\chi_{B(0,r/2)}\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0,r))} \\
&\leq C \|\chi_{B(0,r/2)}\|_{L^{\Phi}(\mathbf{R}^N)} \leq C\{\Phi_{\infty}^{-1}(r^{-N})\}^{-1}.
\end{aligned} \quad (5)$$

Hence

$$\|Mf_0\|_{L^{\Phi(A(0,r))}} \leq Cr^{-N}\{\Phi_{\infty}^{-1}(r^{-N})\}^{-1}.$$

Since $r^{\varepsilon_2-N}\omega(r)\{\Phi_\infty^{-1}(r^{-N})\}^{-1}$ is almost decreasing in $[1, \infty)$ by $(\Phi_\infty\omega 2; -N)$, it follows that

$$\begin{aligned} \int_2^\infty (\omega(r)\|Mf_0\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} &\leq C \int_2^\infty (r^{-N}\omega(r)\{\Phi_\infty^{-1}(r^{-N})\}^{-1})^q \frac{dr}{r} \\ &\leq C. \end{aligned} \quad (6)$$

For $f_{1,r}$, we find for $x \in \mathbf{R}^N \setminus B(0, r)$

$$\begin{aligned} Mf_{1,r}(x) &\leq C|x|^{-N} \int_{B(0,r/2) \setminus B(0,1)} f(y) dy \leq C(|x|/r)^{-N} H_{-N}^0 f(r/2) \\ &\leq C|x|^{-N} r^{N-\varepsilon'_2} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_{1/2}^r (t^{\varepsilon'_2} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \end{aligned} \quad (7)$$

for $0 < \varepsilon'_2 < \varepsilon_2$ by Lemma 8. Hence, using (5), we have

$$\begin{aligned} \int_2^\infty (\omega(r)\|Mf_{1,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} &\leq C \int_2^\infty r^{-\varepsilon'_2 q} \left\{ \int_{1/2}^r (t^{\varepsilon'_2} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right\} \frac{dr}{r} \\ &\leq C \int_{1/2}^\infty (t^{\varepsilon'_2} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \left(\int_t^\infty r^{-\varepsilon'_2 q} \frac{dr}{r} \right) \frac{dt}{t} \\ &\leq C \int_{1/2}^\infty (\omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \leq C. \end{aligned} \quad (8)$$

For $f_{2,r}$, by Lemma 6

$$\|Mf_{2,r}\|_{L^\Phi(A(0,r))} \leq C\|f_{2,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(B(0,4r) \setminus B(0,r/2))},$$

which implies

$$\int_2^\infty (\omega(r)\|Mf_{2,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \leq C. \quad (9)$$

For $f_{3,r}$, we find for $x \in B(0, 2r)$

$$\begin{aligned} Mf_{3,r}(x) &\leq C \int_{\mathbf{R}^N \setminus B(0,4r)} f(y) |y|^{-N} dy \leq CH_0^\infty f(2r) \\ &\leq Cr^{\varepsilon'_1} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon'_1} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \end{aligned} \quad (10)$$

for $0 < \varepsilon'_1 < \varepsilon_1$ by Lemma 7. Hence, by Lemma 4

$$\begin{aligned} \|Mf_{3,r}\|_{L^\Phi(A(0,r))} &\leq \|Mf_{3,r}\|_{L^\Phi(B(0,2r))} \\ &\leq Cr^{\varepsilon'_1}\omega(r)^{-1} \left(\int_r^\infty (t^{-\varepsilon'_1}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q}, \end{aligned}$$

so that

$$\begin{aligned} &\int_2^\infty (\omega(r)\|Mf_{3,r}\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \\ &\leq C \int_2^\infty r^{\varepsilon'_1 q} \left(\int_r^\infty (t^{-\varepsilon'_1}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right) \frac{dr}{r} \\ &\leq C \int_2^\infty (t^{-\varepsilon'_1}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \left(\int_2^t r^{\varepsilon'_1 q} \frac{dr}{r} \right) \frac{dt}{t} \\ &\leq C \int_2^\infty (\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \leq C. \end{aligned} \tag{11}$$

Combining (6), (8), (9) and (11), we obtain (4).

Finally we show

$$\|Mf\|_{L^\Phi(B(0,4))} \leq C$$

since

$$\|Mf\|_{L^\Phi(B(0,2))} + \left(\int_1^2 (\omega(r)\|Mf\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \right)^{1/q} \leq C\|Mf\|_{L^\Phi(B(0,4))}.$$

Set

$$f = f\chi_{B(0,8)} + f\chi_{\mathbf{R}^N \setminus B(0,8)} = f_4 + f_5.$$

By Lemmas 6 and 2,

$$\|Mf_4\|_{L^\Phi(B(0,4))} \leq C\|f\|_{L^\Phi(B(0,8))} \leq C$$

and

$$\begin{aligned} \|Mf_5\|_{L^\Phi(B(0,4))} &= \|Mf_{3,2}\|_{L^\Phi(B(0,4))} \\ &\leq C \left(\int_2^\infty (\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \leq C \end{aligned}$$

by (10). □

4. Boundedness of the Hardy operator

For a locally integrable function f on $\mathbf{R}^N$ and $\beta \in \mathbf{R}$, the Hardy functions $\hat{H}_\beta^\infty f$ and $\hat{H}_\beta^0 f$ are defined by

$$\hat{H}_\beta^\infty f(x) = |x|^\beta \int_{\mathbf{R}^N \setminus (B(0, |x|) \cup B(0, 1))} |y|^{-N-\beta} |f(y)| dy$$

and

$$\hat{H}_\beta^0 f(x) = |x|^\beta \int_{B(0, |x|) \setminus B(0, 1)} |y|^{-N-\beta} |f(y)| dy,$$

respectively.

LEMMA 9. For a real number β , suppose that $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that $(\Phi \omega; \beta) \quad t^{-\varepsilon_3 + \beta} \{\bar{\Phi}^{-1}(0, t^{-N})\}^{-1}$ is almost increasing in $(0, 1]$ for some $\varepsilon_3 > 0$. Then there exists a constant $C > 0$ such that

$$\| |\cdot|^\beta \|_{L^\Phi(B(0, r))} \leq Cr^\beta \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-1}$$

for all $0 < r \leq 1$.

PROOF. Let $0 < r \leq 1$. First note from (1), $(\Phi 2)$ and $(\Phi 3; \infty; q)$ that

$$r^{-N} < |x|^{-N} \leq 2A_1 A_{2, \infty, q} A_3 \bar{\Phi}(0, |x|^{-N/q})$$

for $x \in B(0, r)$, so that we have by Lemma 3 (2) and (4)

$$|x| \leq C \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-q/N}$$

for $x \in B(0, r)$ and

$$\bar{\Phi}^{-1}(0, r^{-N}) \geq \bar{\Phi}^{-1}(0, 1) > 0.$$

Therefore we find by $(\Phi 5; \eta)$ and $\eta \leq q/N$

$$\int_{B(0, r)} \bar{\Phi}(x, \bar{\Phi}^{-1}(0, r^{-N})) dx \leq C \int_{B(0, r)} \bar{\Phi}(0, \bar{\Phi}^{-1}(0, r^{-N})) dx \leq C,$$

so that

$$\|\chi_{B(0, r)}\|_{L^\Phi(\mathbf{R}^N)} \leq C \{\bar{\Phi}^{-1}(0, r^{-N})\}^{-1}.$$

Hence we have by $(\Phi \omega; \beta)$

$$\begin{aligned}
\| |\cdot|^\beta \|_{L^\Phi(B(0,r))} &\leq \sum_{j=1}^{\infty} \| |\cdot|^\beta \|_{L^\Phi(A(0,2^{-j}r))} \\
&\leq C \sum_{j=1}^{\infty} (2^{-j}r)^\beta \{ \bar{\Phi}^{-1}(0, (2^{-j}r)^{-N}) \}^{-1} \\
&\leq Cr^{-\varepsilon_3+\beta} \{ \bar{\Phi}^{-1}(0, r^{-N}) \}^{-1} \sum_{j=1}^{\infty} (2^{-j}r)^{\varepsilon_3} \\
&\leq Cr^\beta \{ \bar{\Phi}^{-1}(0, r^{-N}) \}^{-1},
\end{aligned}$$

as required. $\square$

THEOREM 2. *For a real number β , suppose that $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that $\Phi(x, t)$ satisfies $(\Phi \omega; \beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; \beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_\beta^\infty f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq C \|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$. Let $r \geq 2$. Then we have by Lemmas 4 and 7

$$\begin{aligned}
\|\hat{H}_\beta^\infty f\|_{L^\Phi(A(0,r))} &\leq CH_\beta^\infty f(r/2) \|1\|_{L^\Phi(A(0,r))} \\
&\leq Cr^{\varepsilon'_1} \omega(r)^{-1} \left(\int_{r/2}^{\infty} (t^{-\varepsilon'_1} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q}
\end{aligned}$$

for $0 < \varepsilon'_1 < \varepsilon_1$. Therefore, as in the proof of Theorem 1, we obtain

$$\int_2^{\infty} (\omega(r) \|\hat{H}_\beta^\infty f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \leq C.$$

Finally we show

$$\|\hat{H}_\beta^\infty f\|_{L^\Phi(B(0,4))} \leq C.$$

Note from Lemmas 7 and 2 that

$$\begin{aligned}
\hat{H}_\beta^\infty f(x) &\leq 2^\beta |x|^\beta H_\beta^\infty f(1/2) \\
&\leq C |x|^\beta \left\{ \left(\int_1^{\infty} (\omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q} + \int_{B(0,2) \setminus B(0,1)} f(y) dy \right\} \\
&\leq C |x|^\beta.
\end{aligned}$$

Hence we obtain by Lemma 9

$$\|\hat{H}_\beta^\infty f\|_{L^\Phi(B(0,4))} \leq C \|\cdot\|^\beta_{L^\Phi(B(0,4))} \leq C,$$

as required. $\square$

In the same manner, using Lemma 8 instead of Lemma 7, we can prove the following result.

THEOREM 3. *For a real number β , suppose that $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that $\Phi(x, t)$ satisfies $(\Phi \omega; \beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; \beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_\beta^0 f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq C \|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

In fact, note that

$$\hat{H}_\beta^0 f(x) \leq C |x|^\beta \int_{B(0,4) \setminus B(0,1)} |f(y)| dy \leq C |x|^\beta$$

for $x \in B(0,4)$ and $f \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ with $\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$.

5. Sobolev's inequality

For $0 < \alpha < N$, the Riesz potential $I_\alpha f$ is defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^N} |x - y|^{\alpha-N} f(y) dy$$

for a locally integrable function f on $\mathbf{R}^N$.

LEMMA 10. *Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; -\alpha)$ for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0, 2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,*

$$\begin{aligned} & |I_\alpha(f \chi_{\mathbf{R}^N \setminus B(0, 4r)})(x)| \\ & \leq C r^{\varepsilon+\alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \geq 1$, $x \in B(0, 2r)$ and $0 < \varepsilon < \varepsilon_1$. Note from Lemma 7 with $\beta = -\alpha$ that

$$\begin{aligned}
& |I_\alpha(f\chi_{\mathbf{R}^N \setminus B(0, 4r)})(x)| \\
& \leq C \int_{\mathbf{R}^N \setminus B(0, 4r)} |y|^{\alpha-N} f(y) dy \\
& \leq Cr^{\varepsilon+\alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q},
\end{aligned}$$

as required. $\square$

LEMMA 11. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; -N)$ for $\varepsilon_2 > 0$. Then, for $0 < \varepsilon < \varepsilon_2$, there exists a constant $C > 0$ such that for all $x \in \mathbf{R}^N \setminus B(0, r)$ with $r \geq 1$ and nonnegative functions $f \in L_{loc}^1(\mathbf{R}^N)$,

$$\begin{aligned}
|I_\alpha(f\chi_{B(0, r/2) \setminus B(0, 1)})(x)| & \leq C(|x|/r)^{\alpha-N} r^{-\varepsilon+\alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \\
& \quad \times \left(\int_{1/2}^r (t^\varepsilon \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}.
\end{aligned}$$

PROOF. Let $f \in L_{loc}^1(\mathbf{R}^N)$ be a nonnegative function on $\mathbf{R}^N$. Let $r \geq 1$, $x \in \mathbf{R}^N \setminus B(0, r)$ and $0 < \varepsilon < \varepsilon_2$. Note that

$$\begin{aligned}
|I_\alpha(f\chi_{B(0, r/2) \setminus B(0, 1)})(x)| & \leq C|x|^{\alpha-N} \int_{B(0, r/2) \setminus B(0, 1)} f(y) dy \\
& = C(|x|/r)^{\alpha-N} r^\alpha H_{-N}^0 f(r/2),
\end{aligned}$$

so that Lemma 8 with $\beta = -N$ gives the required result. $\square$

We consider a function

$$\Psi(x, t) = t\psi(x, t) : \mathbf{R}^N \times [0, \infty) \rightarrow [0, \infty)$$

satisfying the conditions $(\Phi 1)$ – $(\Phi 4)$ with ϕ replaced by ψ .

Now we consider the following conditions:

$(\Phi_\infty 2')$ there exists a constant $Q \geq 1$ such that

$$\Phi_\infty(g^*(x)) \leq Q(1 + |x|)^{-N}$$

for all $x \in \mathbf{R}^N$, where $g^*(x) = \max\{g(x), Mg(x)\}$;
 $(\Phi\alpha)$ $r \mapsto r^{\varepsilon+\alpha} \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0, \infty)$ for some $\varepsilon > 0$;

$(\Psi\Phi\alpha)$ there exists a constant $Q \geq 1$ such that

$$\Psi(x, t\Phi(x, t)^{-\alpha/N}) \leq Q\Phi(x, t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

REMARK 10. Let $\Phi(x, t)$ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8 with $\tau = 1$. Assume

$$\inf_{x \in \mathbf{R}^N} (N - \alpha p_1(x)) > 0 \quad \text{and} \quad \inf_{x \in \mathbf{R}^N} (N - \alpha p_2(x)) > 0.$$

Then $\Phi(x, t)$ satisfies $(\Phi\alpha)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$.

Set

$$\Psi(x, t) = \begin{cases} t^{p_1^*(x)} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/t))^{-q_{1,j}(x) p_1^*(x)/p_1(x)} & \text{if } 0 \leq t \leq 1; \\ t^{p_2^*(x)} \prod_{j=1}^{k_2} (L_{e-1}^{(j)}(t))^{q_{2,j}(x) p_2^*(x)/p_2(x)} & \text{if } t \geq 1, \end{cases}$$

where

$$\frac{1}{p_i^*(x)} = \frac{1}{p_i(x)} - \frac{\alpha}{N}$$

for $i = 1, 2$. Then $\Psi(x, t)$ satisfies $(\Psi\Phi\alpha)$.

As Sobolev's inequality for Riesz potentials of functions in Musielak-Orlicz spaces $L^\Phi(\mathbf{R}^N)$, we give the following lemma ([30, Corollary 6.5]). Here we shall state our result without assumptions

$(\Phi_\infty 3)$ $r \mapsto r^\gamma \Phi_\infty^{-1}(r^{-N})$ is almost increasing on $[1, \infty)$ for some $0 < \gamma < N$ and

$(\Phi_\infty \alpha)$ $r \mapsto r^{\varepsilon+\alpha} \Phi_\infty^{-1}(r^{-N})$ is almost decreasing on $[1, \infty)$ for some $\varepsilon > 0$, which are assumed in [30] (see Remark 11 below).

LEMMA 12. Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ holds. Further, assume that $\Psi(x, t)$ satisfies $(\Psi\Phi\alpha)$. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{L^\Psi(\mathbf{R}^N)} \leq C \|f\|_{L^\Phi(\mathbf{R}^N)}$$

for all $f \in L^\Phi(\mathbf{R}^N)$.

REMARK 11. Assumptions $(\Phi 3; 0; p)$ and $(\Phi 3; \infty; q)$ imply $(\Phi_\infty 3)$ and $(\Phi \alpha)$ implies $(\Phi_\infty \alpha)$.

In fact, we see from $(\Phi 3; 0; p)$ and $(\Phi 3; \infty; q)$ that there exist constants $c > 0$ and $\varepsilon > 0$ such that

$$t^{-(1+\varepsilon)} \Phi(x, t) \leq c s^{-(1+\varepsilon)} \Phi(x, s)$$

for all $0 < t < s$, so that for $0 < t < s \leq \max\{1, \Phi_\infty^{-1}(1)\}$, there exists a point $x_0 \in \mathbf{R}^N$ such that $g(x_0) \leq t < s \leq \max\{1, \Phi_\infty^{-1}(1)\}$ and

$$t^{-(1+\varepsilon)} \Phi_\infty(t) \leq Q t^{-(1+\varepsilon)} \Phi(x_0, t) \leq c Q s^{-(1+\varepsilon)} \Phi(x_0, s) \leq c Q^2 s^{-(1+\varepsilon)} \Phi_\infty(s)$$

by $(\Phi_\infty 1)$ and $g \in L^\tau(\mathbf{R}^N)$, where Q is the constant appearing in $(\Phi_\infty 1)$. Since $\Phi_\infty^{-1}(r^{-N}) \leq \Phi_\infty^{-1}(1)$ for all $r \geq 1$, we see that $(\Phi_\infty 3)$ holds with $\gamma = N/(1 + \varepsilon)$. Similarly we can show that $(\Phi\alpha)$ implies $(\Phi_\infty\alpha)$.

Further, as the definition of $\Phi_\infty(t)$, we consider a convex function $\Psi_\infty(t) = t\psi_\infty(t) : [0, \infty) \rightarrow [0, \infty)$ such that $\psi_\infty(t) > 0$ for $t > 0$, $\psi_\infty(t)$ is increasing on $[0, \infty)$ and satisfies the doubling condition and

$(\Psi_\infty 1)$ there exists a constant $Q \geq 1$ such that

$$Q^{-1}\Psi(x, t) \leq \Psi_\infty(t) \leq Q\Psi(x, t) \quad \text{whenever } g(x) \leq t \leq 1,$$

where g is the function appearing in $(\Phi 6; \tau)$.

Now we show the Sobolev type inequality for Riesz potentials of functions in $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

THEOREM 4. *Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi\alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$, $(\Phi_\infty\omega 1; -\alpha)$ and $(\Phi_\infty\omega 2; -N)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{\Psi, q, \omega}(\mathbf{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

REMARK 12. Suppose $\Phi(x, t)$ satisfies $(\Phi\alpha)$ and $\Psi(x, t)$ satisfies $(\Psi\Phi\alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ holds.

To show Theorem 4, we need to verify that the following conditions hold:
 $(\Psi_\infty 2)$ there exists a constant $Q \geq 1$ such that

$$\Psi_\infty(g(x)) \leq Q(1 + |x|)^{-N}$$

for all $x \in \mathbf{R}^N$;

$(\Psi_\infty\Phi_\infty\alpha)$ there exists a constant $Q > 0$ such that

$$\sup_{t \geq 1} t^\alpha \Phi_\infty^{-1}(t^{-N}) \{\Psi_\infty^{-1}(t^{-N})\}^{-1} \leq Q.$$

First we show that $(\Psi_\infty 2)$ holds. Since we have by $(\Psi\Phi\alpha)$ and $(\Psi_\infty 1)$

$$\begin{aligned} \Phi(x, g(x)) &\geq C\Psi(x, g(x))\Phi(x, g(x))^{-\alpha/N} \\ &\geq C\Psi(x, g(x))\Phi(x, 1)^{-\alpha/N} \geq C\Psi(x, g(x)) \geq C\Psi_\infty(g(x)), \end{aligned}$$

we find by $(\Phi_\infty 1)$ and $(\Phi_\infty 2)$

$$\Psi_\infty(g(x)) \leq C\Phi(x, g(x)) \leq C\Phi_\infty(g(x)) \leq C(1 + |x|)^{-N}.$$

Next we show that $(\Psi_\infty \Phi_\infty \alpha)$ holds. For $0 < t \leq \max\{1, \Phi_\infty^{-1}(1)\}$, there exists a point $x_0 \in \mathbf{R}^N$ such that $g(x_0) \leq t \leq \max\{1, \Phi_\infty^{-1}(1)\}$ by $g \in L^\tau(\mathbf{R}^N)$. Then note that

$$t\Phi(x_0, t)^{-\alpha/N} \geq Ct \geq Cg(x_0)$$

and $t\Phi(x_0, t)^{-\alpha/N} \leq C$ by $(\Phi\alpha)$, so that we find by $(\Phi_\infty 1)$ and $(\Psi_\infty 1)$

$$\Psi_\infty(t\Phi_\infty(t)^{-\alpha/N}) \leq C\Phi_\infty(t).$$

Since $\Phi_\infty^{-1}(r^{-N}) \leq \Phi_\infty^{-1}(1)$ for all $r \geq 1$, we obtain that $(\Psi_\infty \Phi_\infty \alpha)$ holds.

PROOF (Proof of Theorem 4). Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$. For $r \geq 2$ set

$$\begin{aligned} f &= f\chi_{B(0,1)} + f\chi_{B(0,r/2) \setminus B(0,1)} + f\chi_{B(0,4r) \setminus B(0,r/2)} + f\chi_{\mathbf{R}^N \setminus B(0,4r)} \\ &= f_0 + f_{1,r} + f_{2,r} + f_{3,r}. \end{aligned}$$

For f_0 , we see that

$$I_\alpha f_0(x) \leq C|x|^{q-N} \int_{B(0,1)} f(y) dy \leq C|x|^{q-N}$$

for $x \in \mathbf{R}^N \setminus B(0, r)$ by Lemma 2. By Lemmas 12 and 4

$$\begin{aligned} r^N \| |\cdot|^{q-N} \|_{L^q(\mathbf{R}^N \setminus B(0,r))} \\ \leq C \| I_\alpha \chi_{B(0,r/2)} \|_{L^q(\mathbf{R}^N \setminus B(0,r))} \\ \leq C \| \chi_{B(0,r/2)} \|_{L^{\Phi}(\mathbf{R}^N)} \leq C \{ \Phi_\infty^{-1}(r^{-N}) \}^{-1}. \end{aligned} \quad (12)$$

Hence

$$\| I_\alpha f_0 \|_{L^q(A(0,r))} \leq Cr^{-N} \{ \Phi_\infty^{-1}(r^{-N}) \}^{-1},$$

and using $(\Phi_\infty \omega 2; -N)$, we have (cf. (6))

$$\int_2^\infty (\omega(r) \| I_\alpha f_0 \|_{L^q(A(0,r))})^q \frac{dr}{r} \leq C. \quad (13)$$

For $f_{1,r}$, by Lemma 11 and (12), we have

$$\| I_\alpha f_{1,r} \|_{L^q(A(0,r))} \leq Cr^{-\varepsilon'_2} \omega(r)^{-1} \left(\int_{1/2}^r (t^{\varepsilon'_2} \omega(t) \| f \|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for $0 < \varepsilon'_2 < \varepsilon_2$, which implies (cf. (8))

$$\int_2^\infty (\omega(r) \| I_\alpha f_{1,r} \|_{L^q(A(0,r))})^q \frac{dr}{r} \leq C. \quad (14)$$

For $f_{2,r}$, we use Lemma 12 and have

$$\|I_{\alpha}f_{2,r}\|_{L^{\Psi}(A(0,r))} \leq C\|f\|_{L^{\Phi}(B(0,4r) \setminus B(0,r/2))}.$$

Hence

$$\int_2^{\infty} (\omega(r)\|I_{\alpha}f_{2,r}\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \leq C. \quad (15)$$

To treat $f_{3,r}$, we remark that Lemma 4 holds for Ψ by $(\Psi_{\infty}2)$ and hence

$$\|1\|_{L^{\Psi}(B(0,r))} \leq C\{\Psi_{\infty}^{-1}(r^{-N})\}^{-1} \leq Cr^{-\alpha}\{\Phi_{\infty}^{-1}(r^{-N})\}^{-1}$$

by $(\Psi_{\infty}\Phi_{\infty}\alpha)$. Thus, we find by Lemma 10

$$\begin{aligned} & \|I_{\alpha}f_{3,r}\|_{L^{\Psi}(A(0,r))} \\ & \leq \|I_{\alpha}f_{3,r}\|_{L^{\Psi}(B(0,r))} \\ & \leq Cr^{\varepsilon'_1}\omega(r)^{-1} \left(\int_r^{\infty} (t^{-\varepsilon'_1}\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \end{aligned} \quad (16)$$

for $0 < \varepsilon'_1 < \varepsilon_1$. It then follows that

$$\int_2^{\infty} (\omega(r)\|I_{\alpha}f_{3,r}\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \leq C. \quad (17)$$

Combining (13), (14), (15) and (17), we obtain

$$\int_2^{\infty} (\omega(r)\|I_{\alpha}f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \leq C.$$

Finally, Lemma 12 and (16) with $r = 2$ yield

$$\begin{aligned} & \|I_{\alpha}f\|_{L^{\Psi}(B(0,2))} + \left(\int_1^2 (\omega(r)\|I_{\alpha}f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \right)^{1/q} \\ & \leq C\|I_{\alpha}f\|_{L^{\Psi}(B(0,4))} \\ & \leq C \left(\|f\|_{L^{\Phi}(B(0,8))} + \int_1^{\infty} (\omega(t)\|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right) \leq C, \end{aligned}$$

which proves the theorem. $\square$

REMARK 13. Let Φ be as in Example 1 and let $\Phi_{\infty}(t)$ be as in Remark 8. If $\omega(r) = r^{\nu}$, then $(\Phi_{\infty}\omega 1; -\alpha)$ and $(\Phi_{\infty}\omega 2; -N)$ hold when

$$\alpha - N/p_1(\infty) < \nu < N(1 - 1/p_1(\infty)).$$

REMARK 14. Let $\Phi(x, t)$, $\Phi_\infty(t)$ and $\Psi(x, t)$ be as in Example 1, Remark 8 and Remark 10. Assume

$$\inf_{x \in \mathbf{R}^N} (N - \alpha p_1(x)) > 0.$$

Set $\bar{\psi}_\infty(t) = \sup_{0 \leq s \leq t} \{s^{p_1^*(\infty)-1} \prod_{j=1}^{k_1} (L_{e-1}^{(j)}(1/s))^{-q_{1,j}(\infty)p_1^*(\infty)/p_1(\infty)}\}$ and

$$\Psi_\infty(t) = \int_0^t \bar{\psi}_\infty(r) \frac{dr}{r},$$

where

$$\frac{1}{p_1^*(\infty)} = \frac{1}{p_1(\infty)} - \frac{\alpha}{N}.$$

Then $\Psi_\infty(t)$ satisfies $(\Psi_\infty 1)$.

6. Sobolev's inequality for the generalized Riesz potential

To obtain general results, for $0 < \alpha < N$ and an integer $k \geq 1$, we define the generalized Riesz potential $I_{\alpha,k}f$ of order α of a locally integrable function f on $\mathbf{R}^N$ by

$$\begin{aligned} I_{\alpha,k}f(x) &= \int_{B(0,1)} I_\alpha(x-y)f(y)dy \\ &+ \int_{\mathbf{R}^N \setminus B(0,1)} \left\{ I_\alpha(x-y) - \sum_{\{\mu: |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y) \right\} f(y)dy, \end{aligned}$$

where $I_\alpha(x) = |x|^{\alpha-N}$ (see [32, 33]).

Set

$$\tilde{I}_{\alpha,k}(x, y) = I_\alpha(x-y) - \sum_{\{\mu: |\mu| \leq k-1\}} \frac{x^\mu}{\mu!} (D^\mu I_\alpha)(-y)$$

and

$$\tilde{I}_{\alpha,k}f(x) = \int_{\mathbf{R}^N \setminus B(0,1)} \tilde{I}_{\alpha,k}(x, y)f(y)dy$$

for a locally integrable function f on $\mathbf{R}^N$.

The following estimates are fundamental (see [32], [33] and [49]).

- LEMMA 13. (1) If $2|x| < |y|$, then $|\tilde{I}_{\alpha,k}(x, y)| \leq C|x|^k|y|^{\alpha-N-k}$.
 (2) If $|x|/2 \leq |y| \leq 2|x|$, then $|\tilde{I}_{\alpha,k}(x, y)| \leq C|x-y|^{\alpha-N}$.
 (3) If $1 \leq |y| \leq |x|/2$, then $|\tilde{I}_{\alpha,k}(x, y)| \leq C|x|^{k-1}|y|^{\alpha-N-(k-1)}$.

LEMMA 14. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; k - \alpha)$ for $\varepsilon_1 > 0$. Then, for $0 < \varepsilon < \varepsilon_1$, there exists a constant $C > 0$ such that, for all $x \in B(0, 2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,

$$\begin{aligned} & |\tilde{I}_{\alpha, k}(f\chi_{\mathbf{R}^N \setminus B(0, 4r)})(x)| \\ & \leq Cr^{\varepsilon + \alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon} \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be nonnegative, $r \geq 1$ and $x \in B(0, 2r)$. By Lemma 13 (1),

$$\begin{aligned} |\tilde{I}_{\alpha, k}(f\chi_{\mathbf{R}^N \setminus B(0, 4r)})(x)| & \leq C|x|^k \int_{\mathbf{R}^N \setminus B(0, 4r)} |y|^{\alpha - N - k} f(y) dy \\ & \leq Cr^\alpha H_{k - \alpha}^\infty f(2r), \end{aligned}$$

so that we obtain the required inequality by Lemma 7. $\square$

LEMMA 15. Assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; k - 1 - \alpha)$ for $\varepsilon_2 > 0$. Then, for $0 < \varepsilon < \varepsilon_2$, there exists a constant $C > 0$ such that for all $x \in B(0, 2r)$ with $r \geq 1$ and nonnegative functions $f \in L^1_{loc}(\mathbf{R}^N)$,

$$\begin{aligned} & |\tilde{I}_{\alpha, k}(f\chi_{B(0, |x|/2)})(x)| \\ & \leq Cr^{-\varepsilon + \alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_{1/2}^r (t^\varepsilon \omega(t) \|f\|_{L^\Phi(A(0, t))})^q \frac{dt}{t} \right)^{1/q}. \end{aligned}$$

PROOF. Let $f \in L^1_{loc}(\mathbf{R}^N)$ be nonnegative, $r \geq 1$ and $x \in B(0, 2r)$. By Lemma 13 (3),

$$\begin{aligned} |\tilde{I}_{\alpha, k}(f\chi_{B(0, |x|/2)})(x)| & \leq C|x|^{k-1} \int_{B(0, |x|/2) \setminus B(0, 1)} |y|^{\alpha - N - (k-1)} f(y) dy \\ & \leq Cr^\alpha H_{k-1-\alpha}^0 f(r), \end{aligned}$$

so that we obtain the required inequality by Lemma 8. $\square$

Now we give the Sobolev type inequality for generalized Riesz potentials of functions in $\mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

THEOREM 5. Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. Assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1;$

$k - \alpha$) and $(\Phi_\infty \omega 2; k - 1 - \alpha)$. Then there exists a constant $C > 0$ such that

$$\|I_{\alpha,k} f\|_{\mathcal{H}^{\Psi,q,\omega}(\mathbf{R}^N)} \leq C \|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)} \leq 1$. For $r \geq 2$ and fixed $x \in A(0, r)$, set

$$\begin{aligned} f &= f\chi_{B(0,1)} + f\chi_{B(0,|x|/2) \setminus B(0,1)} + f\chi_{B(0,4r) \setminus B(0,|x|/2)} + f\chi_{\mathbf{R}^N \setminus B(0,4r)} \\ &= f_0 + f_{1,x} + f_{2,r,x} + f_{3,r}. \end{aligned}$$

For f_0 , we note

$$I_{\alpha,k} f_0(x) = I_{\alpha} f_0(x) \leq C|x|^{\alpha-N}.$$

For $f_{2,r,x}$, by Lemma 13 (1), (2), we see that

$$|I_{\alpha,k} f_{2,r,x}(x)| = |\tilde{I}_{\alpha,k} f_{2,r,x}(x)| \leq C I_{\alpha}(f\chi_{B(0,4r) \setminus B(0,r/2)})(x).$$

Since $I_{\alpha,k} f_{1,x} = \tilde{I}_{\alpha,k}(f\chi_{B(0,|x|/2)})$ and $I_{\alpha,k} f_{3,r} = \tilde{I}_{\alpha,k}(f\chi_{\mathbf{R}^N \setminus B(0,4r)})$, it follows from Lemmas 15 and 14 that

$$\begin{aligned} I_{\alpha,k} f(x) &\leq C \left\{ |x|^{\alpha-N} + I_{\alpha}(f\chi_{B(0,4r) \setminus B(0,r/2)})(x) \right. \\ &\quad + r^{-\varepsilon'_2 + \alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_{1/2}^r (t^{\varepsilon'_2} \omega(t) \|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \\ &\quad \left. + r^{\varepsilon'_1 + \alpha} \omega(r)^{-1} \Phi_\infty^{-1}(r^{-N}) \left(\int_r^\infty (t^{-\varepsilon'_1} \omega(t) \|f\|_{L^{\Phi}(A(0,t))})^q \frac{dt}{t} \right)^{1/q} \right\} \quad (18) \end{aligned}$$

for $x \in A(0, r)$, with $0 < \varepsilon'_1 < \varepsilon_1$ and $0 < \varepsilon'_2 < \varepsilon_2$.

Then, we obtain

$$\int_2^\infty (\omega(r) \|I_{\alpha,k} f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \leq C$$

by the same arguments as in the proof of Theorem 4.

By Lemma 13, we see that $|\tilde{I}_{\alpha,k}(x, y)| \leq C I_{\alpha}(x - y)$ for $|x| \leq 4$ and $|y| \geq 1$. Hence, as in the proof of Theorem 4 we can show that $\|I_{\alpha,k} f\|_{L^{\Psi}(B(0,4))} \leq C$, which implies

$$\|I_{\alpha,k} f\|_{L^{\Psi}(B(0,2))} + \int_1^2 (\omega(r) \|I_{\alpha,k} f\|_{L^{\Psi}(A(0,r))})^q \frac{dr}{r} \leq C. \quad \square$$

REMARK 15. Let Φ be as in Example 1 and let $\Phi_\infty(t)$ be as in Remark 8. If $\omega(r) = r^\nu$, then $(\Phi_\infty\omega 1; k - \alpha)$ and $(\Phi_\infty\omega 2; k - 1 - \alpha)$ hold when

$$\alpha - N/p_1(\infty) - \nu < k < \alpha - N/p_1(\infty) - \nu + 1.$$

7. $\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ and $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$

We further consider the space $\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} = \left(\int_1^\infty (\omega(r) \|f\|_{L^\Phi(B(0, r))})^q \frac{dr}{r} \right)^{1/q} < \infty$$

and the space $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ consisting of all measurable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} = \|f\|_{L^\Phi(B(0, 2))} + \left(\int_1^\infty (\omega(r) \|f\|_{L^\Phi(\mathbf{R}^N \setminus B(0, r))})^q \frac{dr}{r} \right)^{1/q} < \infty.$$

If $\omega(r)$ satisfies

$$(\omega 3) \quad \int_1^\infty \omega(r)^q \frac{dr}{r} < \infty,$$

then

$$L^\Phi(\mathbf{R}^N) = \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \hookrightarrow \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \hookrightarrow \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N) \quad (19)$$

and if ω satisfies

$$(\omega 4) \quad \int_1^\infty \omega(r)^q \frac{dr}{r} = \infty,$$

then

$$\{0\} = \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \subset \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \hookrightarrow \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N) \cap L^\Phi(\mathbf{R}^N).$$

Therefore, it is natural to assume $(\omega 3)$ when we treat the space $\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$; and we assume $(\omega 4)$ when we treat the space $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROPOSITION 1. (1) Suppose $\omega(r)$ satisfies

($\omega 5a$) $r \mapsto r^a \omega(r)$ is almost decreasing on $[1, \infty)$ for some $a > 0$.

Then, $\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) = \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

(2) Suppose $\omega(r)$ satisfies

($\omega 5b$) $r \mapsto r^{-b} \omega(r)$ is almost increasing on $[1, \infty)$ for some $b > 0$.

Then, $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) = \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROOF. (1) Assume $(\omega 5a)$. Let $X = \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$ and $Y = \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$. Since $Y \hookrightarrow X$ in general, we have to show $X \hookrightarrow Y$.

Let $f \in X$ and let K be a compact set in $\mathbf{R}^N$. Then note from Lemma 2 and $(\omega 5a)$ ($(\omega 5a)$ implies $(\omega 3)$) that $\|f\chi_K\|_Y < \infty$. Set $F(r) = \|f\chi_K\|_{L^\phi(A(0,r))}$ and $G(r) = \|f\chi_K\|_{L^\phi(B(0,r))}$. Then

$$G(2r) \leq F(r) + G(r). \quad (20)$$

Set $\omega_*(r) = r^{-a} \inf_{1 \leq s \leq r} s^a \omega(s)$ for $r \geq 1$. Then

$$\omega_*(r) \leq \omega(r) \leq C_0 \omega_*(r) \quad (21)$$

by $(\omega 5a)$ with a constant $C_0 \geq 1$.

Set

$$A = \left(\int_1^\infty (\omega_*(r) F(r))^q \frac{dr}{r} \right)^{1/q},$$

$$B_1 = \left(\int_1^2 (\omega_*(r) G(r))^q \frac{dr}{r} \right)^{1/q} \quad \text{and} \quad B_2 = \left(\int_2^\infty (\omega_*(r) G(r))^q \frac{dr}{r} \right)^{1/q}.$$

All of these are finite values.

By (20)

$$\left(\int_1^\infty (\omega_*(r) G(2r))^q \frac{dr}{r} \right)^{1/q} \leq \begin{cases} A + (B_1^q + B_2^q)^{1/q} & \text{if } q \geq 1, \\ (A^q + B_1^q + B_2^q)^{1/q} & \text{if } 0 < q < 1. \end{cases}$$

Since $r^a \omega_*(r)$ is decreasing, $\omega_*(r/2) \geq 2^a \omega_*(r)$ for $r \geq 2$, so that

$$\int_1^\infty (\omega_*(r) G(2r))^q \frac{dr}{r} = \int_2^\infty (\omega_*(r/2) G(r))^q \frac{dr}{r} \geq 2^{aq} B_2^q.$$

Hence

$$2^a B_2 \leq \begin{cases} A + (B_1^q + B_2^q)^{1/q} & \text{if } q \geq 1, \\ (A^q + B_1^q + B_2^q)^{1/q} & \text{if } 0 < q < 1, \end{cases}$$

which implies

$$(B_1^q + B_2^q)^{1/q} \leq C(A + 2^a B_1) \quad (22)$$

with $C > 0$ depending only on a and q . Note that $B_1 \leq C_1 \|f\chi_K\|_{L^\phi(B(0,2))}$ with $C_1 = \omega(1)(\log 2)^{1/q}$. By (21) $\|f\chi_K\|_Y \leq C_0(B_1^q + B_2^q)^{1/q}$ and $\|f\chi_K\|_X \geq A + \|f\chi_K\|_{L^\phi(B(0,2))}$. Hence (22) implies

$$\|f\chi_K\|_Y \leq C \|f\chi_K\|_X$$

with a constant $C > 0$ independent of K . By the monotone convergence theorem, we obtain the required result.

(2) Assume $(\omega 5b)$. Let X be as above and $Z = \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$. Since $Z \hookrightarrow X$ in general, we shall show $X \hookrightarrow Z$.

Let $f \in X$ and let K be a compact set in $\mathbf{R}^N$. Then note from Lemma 2 that $\|f\chi_K\|_Z < \infty$. Set $F(r) = \|f\chi_K\|_{L^\Phi(A(0, r))}$ and $H(r) = \|f\chi_K\|_{L^\Phi(\mathbf{R}^N \setminus B(0, r))}$. Then

$$H(r) \leq F(r) + H(2r). \quad (23)$$

Set $\omega^*(r) = r^b \sup_{1 \leq s \leq r} s^{-b} \omega(s)$ for $r \geq 1$. Then

$$\omega(r) \leq \omega^*(r) \leq C_2 \omega(r) \quad (24)$$

by $(\omega 5b)$ with a constant $C_2 \geq 1$. Since $r^{-b} \omega^*(r)$ is increasing, $\omega^*(r/2) \leq 2^{-b} \omega^*(r)$ for $r \geq 2$, so that

$$\int_1^\infty (\omega^*(r) H(2r))^q \frac{dr}{r} \leq 2^{-bq} \int_2^\infty (\omega^*(r) H(r))^q \frac{dr}{r}.$$

Hence, by (23), we have

$$\int_1^\infty (\omega^*(r) H(r))^q \frac{dr}{r} \leq C \int_1^\infty (\omega^*(r) F(r))^q \frac{dr}{r},$$

which implies $\|f\chi_K\|_Z \leq C \|f\chi_K\|_X$ in view of (24) with a constant $C > 0$ independent of K . Hence, by the monotone convergence theorem, we obtain the required result. $\square$

The following example shows that there are $\omega(r)$ satisfying $(\omega 3)$ for which $L^\Phi(\mathbf{R}^N) \neq \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \neq \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$; and also there are $\omega(r)$ satisfying $(\omega 4)$ for which $\{0\} \neq \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \neq \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N)$.

EXAMPLE 3. Let $\Phi(x, t) = t^p$, $p \geq 1$ and

$$\omega(r) \sim \log(e + r)^v, \quad v \in \mathbf{R}.$$

(1) If $v < -1/q$, then $\omega(r)$ satisfies $(\omega 3)$ and

$$L^p(\mathbf{R}^N) \neq \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \neq \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N).$$

(2) If $v \geq -1/q$, then $\omega(r)$ satisfies $(\omega 4)$ and

$$\{0\} \neq \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N) \neq \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N).$$

In fact, consider the function

$$f_a(x) = |x|^{-N/p} (\log(e + |x|))^{-a} \chi_{\mathbf{R}^N \setminus B(0, 2)}(x)$$

for $a \in \mathbf{R}$. Then, $\|f_a\|_{L^p(A(0, r))} \sim (\log(e + r))^{-a}$ for $r \geq 2$, so that

$$f_a \in \mathcal{H}^{\Phi, q, \omega}(\mathbf{R}^N) \quad \text{if (and only if) } a > v + 1/q.$$

On the other hand, for $r \geq 3$, $\|f_a\|_{L^p(B(0,r))} \sim (\log(e+r))^{-a+1/p}$ in case $a < 1/p$, $\sim (\log(\log(e+r)))^{1/p}$ in case $a = 1/p$ and $\sim C$ in case $a > 1/p$, so that

$$f_a \in \underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \quad \text{if and only if } a > v + 1/p + 1/q$$

when $v < -1/q$. Thus, $f_a \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus \underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ if $v + 1/q < a \leq v + 1/p + 1/q$ when $v < -1/q$.

Since $f_a \in L^p(\mathbf{R}^N)$ if and only if $ap > 1$, $f_a \in \underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus L^p(\mathbf{R}^N)$ if $v + 1/p + 1/q < a \leq 1/p$. Such a exists when $v < -1/q$.

Next, for $r \geq 2$, $\|f_a\|_{L^p(\mathbf{R}^N \setminus B(0,r))} \sim (\log(e+r))^{-a+1/p}$ in case $a > 1/p$ and $= \infty$ in case $a \leq 1/p$. Hence, in case $v \geq -1/q$, $f_a \in \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N) \setminus \overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ if $v + 1/q < a \leq v + 1/p + 1/q$. Since $\chi_{B(0,1)} \in \overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$, $\overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \neq \{0\}$.

REMARK 16. Since $\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow \mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$, the second inequality (3) also holds with $\|f\|_{\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)}$ replaced by $\|f\|_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$.

Analogous inequality is trivial for $\|f\|_{\overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)}$, since $\overline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N) \hookrightarrow L^\Phi(\mathbf{R}^N)$.

For the boundedness of the maximal operator, we have the following results (cf. [9]).

THEOREM 6. Suppose that $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; 0)$. Then the maximal operator M is bounded from $\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)$ to itself.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathcal{H}}^{\Phi,q,\omega}(\mathbf{R}^N)} \leq 1$. For $r \geq 1$, set

$$f = f\chi_{B(0,2r)} + f\chi_{\mathbf{R}^N \setminus B(0,2r)} = g_{1,r} + g_{2,r}.$$

By Lemma 6,

$$\|Mg_{1,r}\|_{L^\Phi(B(0,r))} \leq C\|g_{1,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(B(0,2r))},$$

so that

$$\int_1^\infty (\omega(r)\|Mg_{1,r}\|_{L^\Phi(B(0,r))})^q \frac{dr}{r} \leq C \int_1^\infty (\omega(r)\|f\|_{L^\Phi(B(0,2r))})^q \frac{dr}{r} \leq C.$$

For $g_{2,r}$, we argue as for $f_{3,r}$ in the proof of Theorem 1 to obtain

$$Mg_{2,r}(x) \leq Cr^{e'_1}\omega(r)^{-1}\Phi_\infty^{-1}(r^{-N})\left(\int_r^\infty (t^{-e'_1}\omega(t)\|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t}\right)^{1/q}$$

for $x \in B(0, r)$ with $0 < \varepsilon'_1 < \varepsilon_1$, which implies

$$\int_1^\infty (\omega(r) \|Mg_{2,r}\|_{L^\Phi(B(0,r))})^q \frac{dr}{r} \leq C \int_1^\infty (\omega(r) \|f\|_{L^\Phi(B(0,2r))})^q \frac{dr}{r} \leq C. \quad \square$$

THEOREM 7. *Suppose that $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; -N)$. Then the maximal operator M is bounded from $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ to itself.*

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$. Then $\|f\|_{L^\Phi(\mathbf{R}^N)} \leq C$.
For $r \geq 2$, set

$$f = f\chi_{B(0,1)} + f\chi_{B(0,r/2) \setminus B(0,1)} + f\chi_{\mathbf{R}^N \setminus B(0,r/2)} = f_0 + f_{1,r} + h_{2,r}.$$

By (5), we see that

$$\|Mf_0\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))} \leq Cr^{-N} \{\Phi_\infty^{-1}(r^{-N})\}^{-1}$$

and using $(\Phi_\infty \omega 2; -N)$, we have

$$\int_2^\infty (\omega(r) \|Mf_0\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C$$

by the same arguments as to obtain (6) in the proof of Theorem 1.

In view of (7) in the proof of Theorem 1, we see

$$\begin{aligned} & \int_2^\infty (\omega(r) \|Mf_{1,r}\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \\ & \leq C \int_{1/2}^\infty (\omega(r) \|f\|_{L^\Phi(A(0,r))})^q \frac{dr}{r} \\ & \leq C \left\{ \|f\|_{L^\Phi(B(0,2))} + \int_1^\infty (\omega(r) \|f\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \right\} \\ & \leq C. \end{aligned}$$

By Lemma 6,

$$\|Mh_{2,r}\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))} \leq C \|h_{2,r}\|_{L^\Phi(\mathbf{R}^N)} = C \|f\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))},$$

so that

$$\begin{aligned} \int_2^\infty (\omega(r) \|Mh_{2,r}\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} & \leq C \int_2^\infty (\omega(r) \|f\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \\ & \leq C. \end{aligned}$$

Thus,

$$\int_2^\infty (\omega(r) \|Mf\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C.$$

Finally, since $\|Mf\|_{L^\Phi(\mathbf{R}^N)} \leq C\|f\|_{L^\Phi(\mathbf{R}^N)} \leq C$ by Lemma 6,

$$\|Mf\|_{L^\Phi(B(0,2))} + \int_1^2 (\omega(r) \|Mf\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C. \quad \square$$

As to Sobolev's inequalities, we have the following results (see also [10]).

THEOREM 8. *Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ and $(\Phi_\infty \omega 1; -\alpha)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{\Psi, q, \omega}(\mathbf{R}^N)} \leq C\|f\|_{\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$. For $r \geq 1$, set

$$f = f\chi_{B(0, 2r)} + f\chi_{\mathbf{R}^N \setminus B(0, 2r)} = g_{1,r} + g_{2,r}.$$

By Lemma 12,

$$\|I_\alpha g_{1,r}\|_{L^\Psi(B(0,r))} \leq C\|g_{1,r}\|_{L^\Phi(\mathbf{R}^N)} = C\|f\|_{L^\Phi(B(0, 2r))},$$

so that

$$\int_1^\infty (\omega(r) \|I_\alpha g_{1,r}\|_{L^\Psi(B(0,r))})^q \frac{dr}{r} \leq C \int_1^\infty (\omega(r) \|f\|_{L^\Phi(B(0, 2r))})^q \frac{dr}{r} \leq C.$$

For $g_{2,r}$, we argue as for $f_{3,r}$ in the proof of Theorem 4 to obtain

$$\|I_\alpha g_{2,r}\|_{L^\Psi(B(0,r))} \leq Cr^{\varepsilon'_1} \omega(r)^{-1} \left(\int_r^\infty (t^{-\varepsilon'_1} \omega(t) \|f\|_{L^\Phi(A(0,t))})^q \frac{dt}{t} \right)^{1/q}$$

for $0 < \varepsilon'_1 < \varepsilon_1$, which implies

$$\int_1^\infty (\omega(r) \|I_\alpha g_{2,r}\|_{L^\Psi(B(0,r))})^q \frac{dr}{r} \leq C \int_1^\infty (\omega(r) \|f\|_{L^\Phi(B(0, 2r))})^q \frac{dr}{r} \leq C. \quad \square$$

THEOREM 9. *Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$ and*

$(\Phi_\infty \omega 2; -N)$ hold. Then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{\Psi, q, \omega}(\mathbf{R}^N)} \leq C \|f\|_{\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$ and for $r \geq 2$, set

$$f = f\chi_{B(0,1)} + f\chi_{B(0,r/2) \setminus B(0,1)} + f\chi_{\mathbf{R}^N \setminus B(0,r/2)} = f_0 + f_{1,r} + h_{2,r}.$$

Since $\int_{B(0,1)} f(y) dy \leq C \|f\|_{L^\Phi(B(0,1))} \leq C$, we see that

$$\|I_\alpha f_0\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))} \leq Cr^{-N} \{\Phi_\infty^{-1}(r^{-N})\}^{-1},$$

and hence

$$\int_2^\infty (\omega(r) \|I_\alpha f_0\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C$$

in the same way as in the proof of Theorem 4.

Also, as in the proof of Theorem 4, we see

$$\int_2^\infty (\omega(r) \|I_\alpha f_{1,r}\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C.$$

For $h_{2,r}$, we use Lemma 12 to obtain

$$\|I_\alpha h_{2,r}\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))} \leq C \|h_{2,r}\|_{L^\Phi(\mathbf{R}^N)} = C \|f\|_{L^\Phi(\mathbf{R}^N \setminus B(0,r/2))},$$

which implies

$$\int_2^\infty (\omega(r) \|I_\alpha h_{2,r}\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C.$$

Finally, since $\|I_\alpha f\|_{L^\Psi(\mathbf{R}^N)} \leq C \|f\|_{L^\Phi(\mathbf{R}^N)} \leq C$,

$$\|I_\alpha f\|_{L^\Psi(B(0,2))} + \int_1^2 (\omega(r) \|I_\alpha f\|_{L^\Psi(\mathbf{R}^N \setminus B(0,r))})^q \frac{dr}{r} \leq C. \quad \square$$

THEOREM 10. Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. For an integer $k \geq 1$, assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1; k - \alpha)$ and $(\Phi_\infty \omega 2; k - 1 - \alpha)$. Then there exists a constant $C > 0$ such that

$$\|I_{\alpha, k} f\|_{\underline{\mathcal{H}}^{\Psi, q, \omega}(\mathbf{R}^N)} \leq C \|f\|_{\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)}$$

for all $f \in \underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$.

PROOF. Let f be a nonnegative measurable function on $\mathbf{R}^N$ such that $\|f\|_{\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)} \leq 1$. Noting that

$$|I_{\alpha, k}(f\chi_{B(0, 4r) \setminus B(0, |x|/2) \setminus B(0, 1)})| \leq CI_{\alpha}(f\chi_{B(0, 4r) \setminus B(0, 1)})$$

for $r \geq 1$ and $|x| < r$, by the same arguments as to obtain (18) in the proof of Theorem 5, we have

$$\begin{aligned} I_{\alpha, k}f(x) \leq C & \left\{ I_{\alpha}(f\chi_{B(0, 4r)})(x) \right. \\ & + r^{-\varepsilon'_2 + \alpha} \omega(r)^{-1} \Phi_{\infty}^{-1}(r^{-N}) \left(\int_{1/2}^r (t^{\varepsilon'_2} \omega(t) \|f\|_{L^{\Phi}(A(0, t))})^q \frac{dt}{t} \right)^{1/q} \\ & \left. + r^{\varepsilon'_1 + \alpha} \omega(r)^{-1} \Phi_{\infty}^{-1}(r^{-N}) \left(\int_r^{\infty} (t^{-\varepsilon'_1} \omega(t) \|f\|_{L^{\Phi}(A(0, t))})^q \frac{dt}{t} \right)^{1/q} \right\} \end{aligned}$$

for $r \geq 1$ and $x \in B(0, r)$, with $0 < \varepsilon'_1 < \varepsilon_1$ and $0 < \varepsilon'_2 < \varepsilon_2$.

Now, by Lemma 12

$$\|I_{\alpha}(f\chi_{B(0, 4r)})\|_{L^{\Psi}(B(0, r))} \leq C\|f\|_{L^{\Phi}(B(0, 4r))}.$$

Thus, in the same way as in the proof of Theorem 4 (with $A(0, r)$ replaced by $B(0, r)$), we obtain

$$\int_1^{\infty} (\omega(r) \|I_{\alpha, k}f\|_{L^{\Psi}(B(0, r))})^q \frac{dr}{r} \leq C. \quad \square$$

8. Variable exponent H-M-M-O spaces

Let $q(r)$ be a measurable function on $[1, \infty)$ satisfying

(Q1) $0 < q^- := \operatorname{ess\,inf}_{r \in [1, \infty)} q(r) \leq \operatorname{ess\,sup}_{r \in [1, \infty)} q(r) =: q^+ < \infty$.

Given $\Phi(x, t)$, $\omega(r)$ and $q(r)$ as above, we denote by $\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$, $\underline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$ and $\overline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$ the classes of locally integrable functions f on $\mathbf{R}^N$ satisfying

$$\|f\|_{\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)} = \|f\|_{L^{\Phi}(B(0, 2))} + \|\omega(\cdot)\|f\|_{L^{\Phi}(A(0, \cdot))}\|_{L^{q(\cdot)}((1, \infty), dr/r)} < \infty,$$

$$\|f\|_{\underline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)} = \|\omega(\cdot)\|f\|_{L^{\Phi}(B(0, \cdot))}\|_{L^{q(\cdot)}((1, \infty), dr/r)} < \infty$$

and

$$\|f\|_{\overline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)} = \|f\|_{L^{\Phi}(B(0, 2))} + \|\omega(\cdot)\|f\|_{L^{\Phi}(\mathbf{R}^N \setminus B(0, \cdot))}\|_{L^{q(\cdot)}((1, \infty), dr/r)} < \infty,$$

respectively, where

$$\|g\|_{L^{q(\cdot)}((1, \infty), dr/r)} = \inf \left\{ \lambda > 0; \int_1^\infty \left(\frac{|g(r)|}{\lambda} \right)^{q(r)} \frac{dr}{r} \leq 1 \right\}.$$

PROPOSITION 2. Suppose $q(r)$ satisfies

(Q2) there exists a constant $q(\infty) \in (0, \infty)$ such that

$$|q(r) - q(\infty)| \leq \frac{C_{q, \infty}}{\log(e + r)}$$

whenever $r \geq 1$ with a constant $C_{q, \infty} \geq 0$.

Then

$$\mathcal{H}^{\Phi, q(\infty), \omega}(\mathbf{R}^N) = \mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N),$$

$$\underline{\mathcal{H}}^{\Phi, q(\infty), \omega}(\mathbf{R}^N) = \underline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$$

and

$$\overline{\mathcal{H}}^{\Phi, q(\infty), \omega}(\mathbf{R}^N) = \overline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N).$$

PROOF. We only prove that $\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N) \subset \mathcal{H}^{\Phi, q(\infty), \omega}(\mathbf{R}^N)$, since the remaining assertions can be proved similarly. Let f be a measurable function on $\mathbf{R}^N$ satisfying $\|f\|_{\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)} \leq 1$. Then note that there exists a constant $c > 0$ such that

$$\int_{1/\sqrt{2}}^\infty (\omega(r) \|f\|_{L^{\Phi}(A(0, r))})^{q(r)} \frac{dr}{r} \leq c.$$

First we show that

$$\omega(r) \|f\|_{L^{\Phi}(A(0, r))} \leq C \quad \text{for } r \geq 1. \quad (25)$$

Let $J(r) = \omega(r) \|f\|_{L^{\Phi}(B(0, \sqrt{2}r) \setminus B(0, r))}$. If $r/\sqrt{2} \leq t \leq r$, then $B(0, \sqrt{2}r) \setminus B(0, r) \subset A(0, t)$, so that

$$J(r) \leq c_1 c_2 \omega(t) \|f\|_{L^{\Phi}(A(0, t))}$$

by $(\omega 2)$. For $r \geq 1$, if $J(r) \geq c_1 c_2$, then

$$c \geq \int_{r/\sqrt{2}}^r (\omega(t) \|f\|_{L^{\Phi}(A(0, t))})^{q(t)} \frac{dt}{t} \geq \frac{\log 2}{2} (c_1^{-1} c_2^{-1} J(r))^{q^-},$$

which implies

$$J(r) \leq c_1 c_2 (2c/\log 2)^{1/q^-}.$$

Therefore,

$$\begin{aligned}\omega(r)\|f\|_{L^\Phi(A(0,r))} &= \omega(r)\|f\|_{L^\Phi(B(0,\sqrt{2}r)\setminus B(0,r))} + \omega(r)\|f\|_{L^\Phi(B(0,2r)\setminus B(0,\sqrt{2}r))} \\ &\leq J(r) + c_1 c_2 J(\sqrt{2}r) \leq C,\end{aligned}$$

which shows (25).

If $r^{-1} < \omega(r)\|f\|_{L^\Phi(A(0,r))}$, then we have by (Q2)

$$(\omega(r)\|f\|_{L^\Phi(A(0,r))})^{q(\infty)} \leq C(\omega(r)\|f\|_{L^\Phi(A(0,r))})^{q(r)}$$

for $r \geq 1$, which gives

$$\begin{aligned}&\int_1^\infty (\omega(r)\|f\|_{L^\Phi(A(0,r))})^{q(\infty)} \frac{dr}{r} \\ &\leq C \int_1^\infty (\omega(r)\|f\|_{L^\Phi(A(0,r))})^{q(r)} \frac{dr}{r} + \int_1^\infty r^{-q(\infty)} \frac{dr}{r} \leq C.\end{aligned}$$

Thus, we obtain the required result. $\square$

By this proposition, Theorems 1, 2, 3, 4 and 5 are valid with $\mathcal{H}^{\Phi,q,\omega}(\mathbf{R}^N)$ replaced by $\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$, provided that $q(r)$ satisfies (Q2), namely we have the following corollaries.

COROLLARY 1. *Assume that $q(r)$ satisfies (Q2). Suppose that $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$ and $(\Phi 6; \tau)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$, $(\Phi_\infty \omega 1; 0)$ and $(\Phi_\infty \omega 2; -N)$. Then the maximal operator M is bounded from $\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$ to itself.*

COROLLARY 2. *Assume that $q(r)$ satisfies (Q2). For a real number β , suppose that $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that $\Phi(x, t)$ satisfies $(\Phi \omega; \beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 1; \beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_\beta^\infty f\|_{\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$.

COROLLARY 3. *Assume that $q(r)$ satisfies (Q2). For a real number β , suppose that $\Phi(x, t)$ satisfies $(\Phi 3; \infty; q)$ and $(\Phi 5; \eta)$ for $q \geq 1$ and $\eta > 0$ satisfying $\eta \leq q/N$. Assume that $\Phi(x, t)$ satisfies $(\Phi \omega; \beta)$ and $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2)$ and $(\Phi_\infty \omega 2; \beta)$. Then there exists a constant $C > 0$ such that*

$$\|\hat{H}_\beta^0 f\|_{\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)} \leq C\|f\|_{\mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi,q(\cdot),\omega}(\mathbf{R}^N)$.

COROLLARY 4. *Assume that $q(r)$ satisfies (Q2). Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. For the function $\Phi_\infty(t)$, assume $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1; -\alpha)$ and $(\Phi_\infty \omega 2; -N)$ hold. Then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{\Psi, q(\cdot), \omega}(\mathbf{R}^N)} \leq C \|f\|_{\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$.

COROLLARY 5. *Assume that $q(r)$ satisfies (Q2). Suppose $\Phi(x, t)$ satisfies $(\Phi 3; 0; p)$, $(\Phi 3; \infty; q)$, $(\Phi 5; \eta)$, $(\Phi 6; \tau)$ and $(\Phi \alpha)$ for $p > 1$, $q > 1$, $\eta > 0$ and $\tau > 0$ satisfying $\eta \leq q/N$ and $\tau \leq p$. Assume that $\Psi(x, t)$ satisfies $(\Psi \Phi \alpha)$. For an integer $k \geq 1$, assume $\Phi_\infty(t)$ satisfies $(\Phi_\infty 2')$, $(\Phi_\infty \omega 1; k - \alpha)$ and $(\Phi_\infty \omega 2; k - 1 - \alpha)$. Then there exists a constant $C > 0$ such that*

$$\|I_{\alpha, k} f\|_{\mathcal{H}^{\Psi, q(\cdot), \omega}(\mathbf{R}^N)} \leq C \|f\|_{\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)}$$

for all $f \in \mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$.

Also, Theorems 6, 8 and 10 hold with $\underline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ replaced by $\mathcal{H}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$, and Theorems 7 and 9 hold with $\overline{\mathcal{H}}^{\Phi, q, \omega}(\mathbf{R}^N)$ replaced by $\overline{\mathcal{H}}^{\Phi, q(\cdot), \omega}(\mathbf{R}^N)$, when $q(r)$ satisfies (Q2).

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