

On a conjecture of Li and Yang

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(Received May 7, 2022)

(Revised October 1, 2022)

ABSTRACT. We study the uniqueness problem of an entire function f when it shares two small functions with its derivative $f^{(k)}$ ($k \geq 1$). This confirms the conjecture posed by Li and Yang [5].

1. Introduction and main result

Let $\mathcal{M}(\mathbb{C})$ be the family of all non-constant functions which are meromorphic in $\mathbb{C}$, whereas $\mathcal{E}(\mathbb{C})$ denotes the family of all non-constant entire functions in $\mathbb{C}$. On the other hand, we denote by $\mathcal{M}_T(\mathbb{C})$ and $\mathcal{E}_T(\mathbb{C})$ the family of all transcendental meromorphic functions in $\mathbb{C}$ and the family of all transcendental entire functions in $\mathbb{C}$ respectively. In this paper, for $f \in \mathcal{M}(\mathbb{C})$, we shall use the standard notations of Nevanlinna's value distribution theory such as $T(r, f), m(r, f), N(r, f), \bar{N}(r, f), \dots$ (see e.g., [4, 10]). We adopt the standard notation $S(r, f)$ for any quantity satisfying the relation $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ except possibly a set of finite linear measure. Let $a, f \in \mathcal{M}(\mathbb{C})$. Then a is said to be a small function of f if $T(r, a) = S(r, f)$. Denote by $\mathcal{S}(f)$ the family of all small functions of $f \in \mathcal{M}(\mathbb{C})$. Let $a \in \mathcal{S}(f) \cap \mathcal{S}(g)$. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM and if we do not consider the multiplicities, then we say that f and g share a IM.

Rubel and Yang [9] were the first to study entire functions that share values with their derivatives. In 1977, they proved if $f \in \mathcal{E}(\mathbb{C})$ shares two finite distinct values CM with f' , then $f \equiv f'$. This result has been generalized from sharing values CM to IM by Mues and Steinmetz [8] and in the case when both shared values are non-zero independently by Gundersen [3]. Since then the subject of sharing values between a meromorphic function and its derivative has been extensively studied by many researchers and a lot of interesting results have been obtained (see [10]).

2020 *Mathematics Subject Classification.* Primary 30D35, Secondary 34A20.

Key words and phrases. Entire function, meromorphic function, small function, Nevanlinna theory, uniqueness, derivative.

In 1991, G. Frank [2] proposed the following conjecture: If $f \in \mathcal{E}(\mathbb{C})$ shares two finite values IM with its k -th derivative ($k \geq 1$), then $f \equiv f^{(k)}$.

In 2000, Li and Yang [5] fully resolved **Frank's conjecture** in the following form.

THEOREM 1.1 ([5]). *Let $f \in \mathcal{E}(\mathbb{C})$ and $a_1, a_2 \in \mathbb{C}$ be distinct. If f and $f^{(k)}$ ($k \geq 1$) share a_1 and a_2 IM, then $f \equiv f^{(k)}$.*

At the end of the paper, Li and Yang [5] gave rise to the following conjecture:

CONJECTURE 1.2. *Theorem 1.1 still holds when a_1 and a_2 are two arbitrary distinct small functions of f .*

To the knowledge of authors, **Conjecture 1.2** is not still confirmed. In this paper, we settle **Conjecture 1.2** at the cost of considering the fact that $a_1'' \not\equiv a_2''$. We now state our main result as follows.

THEOREM 1.3. *Let $f \in \mathcal{E}(\mathbb{C})$ and $a_1, a_2 \in \mathcal{S}(f)$ be non-constant such that $a_1, a_2 \not\equiv \infty$ and $a_1'' \not\equiv a_2''$. If f and $f^{(k)}$ ($k \geq 1$) share a_1 and a_2 IM, then $f \equiv f^{(k)}$.*

REMARK 1.4. *The following example asserts that condition “ $a_1, a_2 \not\equiv \infty$ ” is sharp in Theorem 1.3.*

EXAMPLE 1.1. *Let $f(z) = c + e^{ce^z}$ and $a_1(z) = \frac{c^2}{c - e^{-z}}$, where $c \in \mathbb{C} \setminus \{0\}$. If $a_2 = \infty$, then f and f' certainly share a_2 CM. On the other hand, we see that f and f' also share a_1 CM, but $f \not\equiv f'$.*

First of all, we generalize the definition of IM to IM*. Let $f, g \in \mathcal{M}(\mathbb{C})$ and $a \in \mathcal{S}(f) \cap \mathcal{S}(g)$. Denote by $\bar{N}_0(r, a)$ the counting function of all common zeroes of $f - a$ and $g - a$ ignoring multiplicities. If $\bar{N}(r, a; f) + \bar{N}(r, a; g) - 2\bar{N}_0(r, a) = S(r, f) + S(r, g)$, then we say f and g share a IM*. One can easily prove that Theorem 1.3 is still valid if condition “IM” is replaced by “IM*”.

REMARK 1.5. *We can easily see that Theorem 1.3 is still valid for any $f \in \mathcal{M}(\mathbb{C})$ satisfying $N(r, f) = S(r, f)$.*

REMARK 1.6. *The following examples assert that Theorems A and 1.3 do not hold for any $f \in \mathcal{M}(\mathbb{C})$ satisfying $N(r, f) \neq S(r, f)$.*

EXAMPLE 1.2. Let $f(z) = \frac{4}{1-3e^{-2z}}$. Clearly $N(r, f) \neq S(r, f)$. Note that $f'(z) = \frac{-24e^{-2}}{(1-3e^{-2z})^2}$ and so f and f' share 0 CM. On the other hand, we see that f and f' share 2 IM, but $f \not\equiv f'$.

EXAMPLE 1.3. Let $a(z) = -\frac{1}{3}e^{-2z} + ce^{-z}$, $b(z) = -\frac{1}{3}e^{-2z} - ce^{-z}$ and $h(z) = e^{2ce^z}$, where $c \in \mathbb{R} \setminus \{0\}$. Define $f(z) = b(z) + \frac{b(z)-a(z)}{h(z)-1}$. Let $a_1(z) = b'(z)$ and $a_2(z) = a'(z)$. Clearly, $a_1, a_2 \in \mathcal{S}(f)$ and $N(r, f) \neq S(r, f)$. Also, we deduce that $f'(z) - a_1(z) = e^{2z}(f(z) - a_1(z))(f(z) - b(z))$ and $f'(z) - a_2(z) = e^{2z}(f(z) - a_2(z))(f(z) - a(z))$. Clearly, f and f' share a_1 and a_2 IM, but $f \not\equiv f'$.

EXAMPLE 1.4 ([6]). Let $a, b \in \mathbb{C}$ such that $a - b = \sqrt{2}i$ and w be a non-constant solution of the Riccati differential equation $w' = (w - a_1)(w - a_2)$. Let $f(z) = (w(z) - a)(w(z) - b) - \frac{1}{3}$. Then $w, f \in \mathcal{M}_T(\mathbb{C})$ and $w' \neq 0$. It is easy to verify that $f'' = 6w'f$ and $f'' + \frac{1}{6} = 6(f + \frac{1}{6})^2$. Clearly f and f'' share 0 CM and $-\frac{1}{6}$ IM, but $f \not\equiv f''$.

After considering Theorem 1.3, we ask The following open question:

Open problem. Is it possible to establish Theorem 1.3 without the hypothesis $a_1' \not\equiv a_2''$?

2. Auxiliary lemmas

LEMMA 2.1 ([1]). Let $f \in \mathcal{M}_T(\mathbb{C})$ such that $f^n P(f) = Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n . Then $m(r, P) = S(r, f)$.

LEMMA 2.2 ([11]). If $f, g \in \mathcal{M}(\mathbb{C})$, then

$$N\left(r, \frac{f}{g}\right) - N\left(r, \frac{g}{f}\right) = N(r, f) + N(r, 0; g) - N(r, g) - N(r, 0; f).$$

LEMMA 2.3. Let $f \in \mathcal{E}(\mathbb{C})$ and $a_1, a_2 \in \mathcal{S}(f)$ such that $a_1, a_2 \not\equiv 0, \infty$ and $a_1 \not\equiv a_2$. Suppose

$$\Delta(f) = \begin{vmatrix} f - a_1 & a_1 - a_2 \\ f' - a_1' & a_1' - a_2' \end{vmatrix} = \begin{vmatrix} f - a_2 & a_1 - a_2 \\ f' - a_2' & a_1' - a_2' \end{vmatrix}.$$

Then

- (1) $\Delta(f) \not\equiv 0$,
- (2) $m\left(r, \frac{\Delta(f)}{f - a_i}\right) = S(r, f)$ ($i = 1, 2$),

$$(3) \quad m\left(r, \frac{\Delta(f)}{(f-a_1)(f-a_2)}\right) = S(r, f),$$

$$(4) \quad m\left(r, \frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)}\right) = S(r, f), \text{ where } \beta \in \mathcal{S}(f),$$

$$(5) \quad m\left(r, \frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)}\right) = S(r, f).$$

PROOF. (1) Suppose to the contrary that $\Delta(f) \equiv 0$. Then clearly we have $\frac{f'-a'_1}{f-a_1} \equiv \frac{a'_1-a'_2}{a_1-a_2}$. On integration, we have $f \equiv a_1 + a_0(a_1 - a_2)$, where $a_0 \in \mathbb{C} \setminus \{0\}$. This shows that $f \in \mathcal{S}(f)$, which is a contradiction. Hence $\Delta(f) \not\equiv 0$.

(2) Note that for $i = 1, 2$, we have $\Delta(f) = (a'_1 - a'_2)(f - a_i) - (a_1 - a_2)(f' - a'_i)$, i.e., $\frac{\Delta(f)}{f-a_i} = a'_1 - a'_2 - (a_1 - a_2) \frac{f'-a'_i}{f-a_i}$. Consequently

$$\begin{aligned} m\left(r, \frac{\Delta(f)}{f-a_i}\right) &\leq m(r, a'_1 - a'_2) + m(r, a_1 - a_2) + m\left(r, \frac{f'-a'_i}{f-a_i}\right) + \log 2 \\ &= S(r, f), \end{aligned}$$

for $i = 1, 2$ and so (2) holds.

(3) We see that $\frac{\Delta(f)}{(f-a_1)(f-a_2)} = \frac{1}{a_1-a_2} \left[\frac{\Delta(f)}{f-a_1} - \frac{\Delta(f)}{f-a_2} \right]$. Now (3) follows directly from (2).

(4) We see that $\frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)} = \frac{\Delta(f)}{f-a_1} + \frac{(a_2-\beta)\Delta(f)}{(f-a_1)(f-a_2)}$ and so

$$\begin{aligned} m\left(r, \frac{\Delta(f)(f-\beta)}{(f-a_1)(f-a_2)}\right) &\leq m\left(r, \frac{\Delta(f)}{f-a_1}\right) + m\left(r, \frac{\Delta(f)}{(f-a_1)(f-a_2)}\right) \\ &\quad + m(r, a_2 - \beta). \end{aligned}$$

Now (4) follows directly from (2) and (3).

(5) We see that $m\left(r, \frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)}\right) \leq m\left(r, \frac{\Delta(f)f}{(f-a_1)(f-a_2)}\right) + m\left(r, 1 - \frac{f^{(k)}}{f}\right)$. Now (5) follows directly from (4).

LEMMA 2.4 ([7]). Let $f \in \mathcal{M}(\mathbb{C})$ and $R(f) = \frac{P(f)}{Q(f)}$, where $P(f) = \sum_{k=0}^p a_k f^k$ and $Q(f) = \sum_{j=0}^q b_j f^j$ are two mutually prime polynomials in f . If $a_k, b_j \in \mathcal{S}(f)$ such that $a_p \not\equiv 0$ and $b_q \not\equiv 0$, then $T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f)$.

LEMMA 2.5 ([4]). Let $f \in \mathcal{M}(\mathbb{C})$ and $a_1, a_2 \in \mathcal{S}(f)$. Then $T(r, f) \leq \bar{N}(r, f) + \sum_{i=1}^2 \bar{N}(r, a_i; f) + S(r, f)$.

Henceforth for $f \in \mathcal{E}(\mathbb{C})$, we define the following auxiliary functions

$$\phi = \frac{\Delta(f)(f-f^{(k)})}{(f-a_1)(f-a_2)}, \quad (2.1)$$

$$\psi = \frac{\Delta(f^{(k)})(f - f^{(k)})}{(f^{(k)} - a_1)(f^{(k)} - a_2)}, \quad (2.2)$$

$$H_{nm} = n\phi - m\psi, \quad (2.3)$$

where $m, n \in \mathbb{N}$ and

$$H = \frac{\Delta(f)}{(f - a_1)(f - a_2)} - \frac{\Delta(f^{(k)})}{(f^{(k)} - a_1)(f^{(k)} - a_2)}. \quad (2.4)$$

Differentiating twice, we obtain from (2.1) that

$$\begin{aligned} & ((a_1'' - a_2'')(f - a_1) - (a_1 - a_2)(f'' - a_1''))(f - f^{(k)}) \\ & + ((a_1' - a_2')(f - a_1) - (a_1 - a_2)(f' - a_1'))(f' - f^{(k+1)}) \\ & = \phi'(f - a_1)(f - a_2) + \phi(f' - a_1')(f - a_2) + \phi(f - a_1)(f' - a_2') \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} & ((a_1''' - a_2''')(f - a_1) + (a_1'' - a_2'')(f' - a_1') - (a_1' - a_2')(f'' - a_1'')) \\ & - (a_1 - a_2)(f''' - a_1'''))(f - f^{(k)}) \\ & + 2((a_1'' - a_2'')(f - a_1) - (a_1 - a_2)(f'' - a_1''))(f' - f^{(k+1)}) \\ & + ((a_1' - a_2')(f - a_1) - (a_1 - a_2)(f' - a_1'))(f'' - f^{(k+2)}) \\ & = \phi''(f - a_1)(f - a_2) + 2\phi'(f' - a_1')(f - a_2) + 2\phi'(f - a_1)(f' - a_2') \\ & + \phi(f'' - a_1'')(f - a_2) + 2\phi(f' - a_1')(f' - a_2') \\ & + \phi(f - a_1)(f'' - a_2''). \end{aligned} \quad (2.6)$$

DEFINITION 2.6. Let $k, m, n \in \mathbb{N}$ and $a_1, a_2 \in \mathcal{S}(f)$. Denote by $S_{(m,n)}(a_1)$ the set of those points $z \in \mathbb{C}$ such that z is an a_1 -point of f of order m and an a_1 -point of $f^{(k)}$ of order n . The set $S_{(m,n)}(a_2)$ can be defined similarly. Let $\bar{N}_{(m,n)}(r, a_1; f)$ denote the reduced counting function of f with respect to the set $S_{(m,n)}(a_1)$. Similarly $\bar{N}_{(m,n)}(r, a_2; f)$ denotes the reduced counting function of f with respect to the set $S_{(m,n)}(a_2)$.

Let $z_{p,q} \in S_{(p,q)}(a_1)$ such that $\phi(z_{p,q}) \neq 0, \infty$ and $a_1(z_{p,q}) - a_2(z_{p,q}) \neq 0, \infty$. Then in some neighbourhood of $z_{p,q}$, we get by Taylor's expansion

$$\begin{cases} f(z) - a_1(z) = b_p(z - z_{p,q})^p + b_{p+1}(z - z_{p,q})^{p+1} + \cdots & (b_p \neq 0), \\ f^{(k)}(z) - a_1(z) = c_q(z - z_{p,q})^q + c_{q+1}(z - z_{p,q})^{q+1} + \cdots & (c_q \neq 0), \\ \phi(z) = d_0 + d_1(z - z_{p,q}) + d_2(z - z_{p,q})^2 + \cdots & (d_0 \neq 0). \end{cases} \quad (2.7)$$

Clearly

$$\left\{ \begin{array}{l} f'(z) - a_1'(z) = pb_p(z - z_{p,q})^{p-1} + (p+1)b_{p+1}(z - z_{p,q})^p + \dots, \\ f''(z) - a_1''(z) = p(p-1)b_p(z - z_{p,q})^{p-2} \\ \quad + p(p+1)b_{p+1}(z - z_{p,q})^{p-1} + \dots, \\ f^{(k+1)}(z) - a_1^{(k+1)}(z) = qc_q(z - z_{p,q})^{q-1} + (q+1)c_{q+1}(z - z_{p,q})^q + \dots, \\ f^{(k+2)}(z) - a_1^{(k+2)}(z) = q(q-1)c_q(z - z_{p,q})^{q-2} \\ \quad + (q+1)qc_{q+1}(z - z_{p,q})^{q-1} + \dots \\ \phi'(z) = d_1 + 2d_2(z - z_{p,q}) + 3d_2(z - z_{p,q})^2 \dots, \\ \phi''(z) = 2d_2 + 6d_2(z - z_{p,q}) \dots \end{array} \right. \quad (2.8)$$

Now from (2.7) and (2.8), we see that $z_{p,q}$ is a zero of $\Delta(f(z))$ of multiplicity $p-1$ and

$$f(z) - f^{(k)}(z) = \begin{cases} b_p(z - z_{p,q})^p + \dots, & \text{if } p < q \\ -c_q(z - z_{p,q})^q - \dots, & \text{if } p > q \\ (b_p - c_p)(z - z_{p,q})^p + \dots, & \text{if } p = q. \end{cases} \quad (2.9)$$

If $z_{p,1} \in S_{(p,1)}(a_1)$ ($p \geq 2$) such that $\phi(z_{p,1}) \neq 0, \infty$ and $a_1(z_{p,1}) - a_2(z_{p,1}) \neq 0, \infty$, then from (2.5), (2.7)–(2.9), one can easily conclude that

$$d_0 = pc_1, \quad \text{i.e., } f^{(k+1)}(z_{p,1}) - a_1'(z_{p,1}) = c_1 = \frac{\phi(z_{p,1})}{p}. \quad (2.10)$$

Let $\hat{z}_{p,q} \in S_{(p,q)}(a_2)$ such that $\phi(\hat{z}_{p,q}) \neq 0, \infty$ and $a_1(\hat{z}_{p,q}) - a_2(\hat{z}_{p,q}) \neq 0, \infty$.

Then in some neighbourhood of $\hat{z}_{p,q}$, we get by Taylor's expansion

$$\left\{ \begin{array}{l} f(z) - a_2(z) = \hat{b}_p(z - \hat{z}_{p,q})^p + \hat{b}_{p+1}(z - \hat{z}_{p,q})^{p+1} + \dots \quad (\hat{b}_p \neq 0), \\ f^{(k)}(z) - a_2(z) = \hat{c}_q(z - \hat{z}_{p,q})^q + \hat{c}_{q+1}(z - \hat{z}_{p,q})^{q+1} + \dots \quad (\hat{c}_q \neq 0), \\ \phi(z) = \hat{d}_0 + \hat{d}_1(z - \hat{z}_{p,q}) + \hat{d}_2(z - \hat{z}_{p,q})^2 + \dots \quad (\hat{d}_0 \neq 0). \end{array} \right. \quad (2.11)$$

Similarly if $\hat{z}_{p,1} \in S_{(p,1)}(a_2)$ ($p \geq 2$) such that $\phi(\hat{z}_{p,1}) \neq 0, \infty$ and $a_1(\hat{z}_{p,1}) - a_2(\hat{z}_{p,1}) \neq 0, \infty$, then we immediately get

$$f^{(k+1)}(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1}) = \hat{c}_1 = -\frac{\phi(\hat{z}_{p,1})}{p}. \quad (2.12)$$

LEMMA 2.7. Let $f \in \mathcal{E}(\mathbb{C})$ and $a_1, a_2 \in \mathcal{S}(f)$ such that $a_1, a_2 \not\equiv \infty$ and $a_1 \not\equiv a_2$. If f and $f^{(k)}$ share a_1, a_2 IM and $T(r, f) = T(r, f^{(k)}) + S(r, f)$, then $f \equiv f^{(k)}$.

PROOF. Suppose to the contrary that $f \not\equiv f^{(k)}$. Since f and $f^{(k)}$ share a_1 and a_2 IM, we have

$$\begin{aligned}
\bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) &\leq N(r, 0; f - f^{(k)}) + S(r, f) \\
&\leq T(r, f - f^{(k)}) + S(r, f) \\
&\leq m(r, f - f^{(k)}) + S(r, f) \\
&\leq m(r, f) + m\left(r, 1 - \frac{f^{(k)}}{f}\right) + S(r, f) \\
&\leq T(r, f) + S(r, f).
\end{aligned} \tag{2.13}$$

Also using Lemma 2.5, we have $T(r, f) \leq \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f)$. Therefore we conclude that

$$T(r, f) = \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f). \tag{2.14}$$

Again from Lemma 2.3, we see that $\Delta(f) \not\equiv 0$ and so $\phi \not\equiv 0$. If possible suppose that $\Delta(f^{(k)}) \equiv 0$. Then clearly we have

$$\frac{f^{(k+1)} - a'_1}{f^{(k)} - a_1} \equiv \frac{a'_1 - a'_2}{a_1 - a_2}.$$

On integration, we have $f^{(k)} \equiv a_1 + a_0(a_1 - a_2)$, where $a_0 \in \mathbb{C} \setminus \{0\}$. This shows that $f^{(k)} \in \mathcal{S}(f)$. Since $T(r, f) = T(r, f^{(k)}) + S(r, f)$, it follows that $f \in \mathcal{S}(f)$ which is a contradiction. Therefore $\Delta(f^{(k)}) \not\equiv 0$ and so $\psi \not\equiv 0$. Let $z_{p,q} \in S_{(p,q)}(a_1)$ such that $a_1(z_{p,q}), a_1(z_{p,q}) - a_2(z_{p,q}) \neq 0, \infty$ and $a'_1(z_{p,q}) - a'_2(z_{p,q}) \neq 0$. Then from (2.1) and (2.9), we conclude that $z_{p,q}$ is a zero of $\phi(z)$ of multiplicity $t - 1$, where $t \geq \min\{p, q\} \geq 1$ and so ϕ is holomorphic at $z_{p,q}$.

Let $\hat{z}_{p,q} \in S_{(p,q)}(a_2)$ such that $a_1(\hat{z}_{p,q}), a_1(\hat{z}_{p,q}) - a_2(\hat{z}_{p,q}) \neq 0, \infty$ and $a'_1(\hat{z}_{p,q}) - a'_2(\hat{z}_{p,q}) \neq 0$. Then, in the same way as above, one can easily prove that ϕ is also holomorphic at $\hat{z}_{p,q}$. As a result we have $N(r, \phi) = S(r, f)$. Also from Lemma 2.3, we get $m(r, \phi) = S(r, f)$ and so $\phi \in \mathcal{S}(f)$.

Denote by $\bar{N}(r, a_1; f, f^{(k)} | \geq 2)$ the reduced counting function of common multiple 0-points of $f - a_1$ and $f^{(k)} - a_1$. Since $\phi \in \mathcal{S}(f)$, it follows that

$$\bar{N}(r, a_1; f, f^{(k)} | \geq 2) \leq N(r, 0; \phi) \leq S(r, f). \tag{2.15}$$

Similarly we have

$$\bar{N}(r, a_2; f, f^{(k)} | \geq 2) = S(r, f). \tag{2.16}$$

Denote by $N(r, 0; f - f^{(k)} | f \neq a_1, a_2)$ the counting function of those 0-points of $f - f^{(k)}$ which are neither the 0-points of $f - a_1$ nor the 0-points of $f - a_2$. We denote by $\bar{N}_{(s+1)}(r, 0; f - f^{(k)} | f = a_1, a_2)$ the reduced counting

function of those 0-points of $f - f^{(k)}$ with multiplicity greater than s which are the 0-points of both $f - a_1$ and $f - a_2$.

Now from (2.1), we can easily deduce that

$$\begin{aligned} & \bar{N}_{(2)}(r, 0; f - f^{(k)} | f = a_1, a_2) + N(r, 0; \Delta(f)(f - f^{(k)}) | f \neq a_1, a_2) \\ &= S(r, f). \end{aligned} \quad (2.17)$$

Let $a_3 = a_1 + l(a_1 - a_2)$, where " $l \in \mathbb{N}$ ", and let

$$F = \frac{f - a_1}{a_2 - a_1}. \quad (2.18)$$

Clearly $a_3 \not\equiv a_1, a_2$ and $T(r, F) = T(r, f) + S(r, f)$. Now using the second fundamental theorem and (2.14), we have

$$\begin{aligned} 2T(r, f) &= 2T(r, F) + S(r, f) \\ &\leq \bar{N}(r, \infty; F) + \bar{N}(r, 0; F) + \bar{N}(r, 1; F) + \bar{N}(r, -l; F) + S(r, f) \\ &\leq \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + \bar{N}(r, a_3; f) + S(r, f) \\ &\leq 2T(r, f) - m(r, a_3; f) + S(r, f), \end{aligned}$$

i.e., $m(r, a_3; f) = S(r, f)$. Also from (2.1), we see that

$$\frac{1}{f} = \frac{\Delta(f)}{\phi(f - a_1)(f - a_2)} \left(1 - \frac{f^{(k)}}{f} \right)$$

and so using Lemma 2.3, we get $m(r, 0; f) = S(r, f)$. Therefore we have

$$m(r, 0; f) = S(r, f) \quad \text{and} \quad m(r, a_3; f) = S(r, f), \quad (2.19)$$

where $a_3 = a_1 + l(a_1 - a_2)$, and $l \in \mathbb{N}$. Let $G = \frac{f^{(k)} - a_1}{a_2 - a_1}$. Clearly $T(r, G) = T(r, f^{(k)}) + S(r, f)$. Note that f and $f^{(k)}$ share a_1, a_2 IM and $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Now from (2.14) and using the second fundamental theorem, we have

$$\begin{aligned} 2T(r, f^{(k)}) &= 2T(r, G) + S(r, f) \\ &\leq \bar{N}(r, \infty; G) + \bar{N}(r, 0; G) + \bar{N}(r, 1; G) + \bar{N}(r, -l; G) + S(r, f) \\ &\leq \bar{N}(r, a_1; f^{(k)}) + \bar{N}(r, a_2; f^{(k)}) + \bar{N}(r, a_3; f^{(k)}) + S(r, f) \\ &\leq \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f) \\ &= T(r, f) + T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f) \end{aligned}$$

$$\begin{aligned}
&= 2T(r, f^{(k)}) - m(r, a_3; f^{(k)}) + S(r, f), \\
&\text{i.e., } m(r, a_3; f^{(k)}) = S(r, f).
\end{aligned} \tag{2.20}$$

Again from (2.19) and (2.20), we have

$$\begin{aligned}
m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) &= m\left(r, \frac{f^{(k)} - a_3^{(k)} + a_3^{(k)} - a_3}{f - a_3}\right) \leq m(r, a_3; f) + S(r, f) \\
&= S(r, f), \\
&\text{i.e., } m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) = S(r, f).
\end{aligned} \tag{2.21}$$

Since $T(r, f) = T(r, f^{(k)}) + S(r, f)$, from Lemma 2.2, (2.19) and (2.21), we get

$$\begin{aligned}
m\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) &= T\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) \\
&= T\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + O(1) \\
&= N\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) + m\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) \\
&\quad - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + O(1) \\
&= N\left(r, \frac{f^{(k)} - a_3}{f - a_3}\right) - N\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + S(r, f) \\
&= N(r, a_3; f) - N(r, a_3; f^{(k)}) + S(r, f) \\
&= N(r, a_3; f) + m(r, a_3; f) \\
&\quad - \{N(r, a_3; f^{(k)}) + m(r, a_3; f^{(k)})\} + S(r, f) \\
&= T(r, f) - T(r, f^{(k)}) + S(r, f) = S(r, f), \\
&\text{i.e., } m\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) = S(r, f).
\end{aligned} \tag{2.22}$$

Since f and $f^{(k)}$ share a_1, a_2 IM, we can easily see that $N(r, \psi) = S(r, f)$. Note that

$$\psi = \frac{A(f^{(k)})(f^{(k)} - a_3)}{(f^{(k)} - a_1)(f^{(k)} - a_2)} \left(\frac{f - a_3}{f^{(k)} - a_3} - 1 \right)$$

and so using Lemma 2.3 and (2.22), we have

$$\begin{aligned}
m(r, \psi) &\leq m\left(r, \frac{\Delta(f^{(k)})(f^{(k)} - a_3)}{(f^{(k)} - a_1)(f^{(k)} - a_2)}\right) + m\left(r, \frac{f - a_3}{f^{(k)} - a_3}\right) + O(1) \\
&\leq S(r, f),
\end{aligned}$$

i.e., $m(r, \psi) = S(r, f)$. Consequently, $\psi \in \mathcal{S}(f)$. We now consider the following two cases.

Case 1. Suppose that $H_{nm} \equiv 0$. Then from (2.1) and (2.2), we have

$$n\left(\frac{f' - a'_1}{f - a_1} - \frac{f' - a'_2}{f - a_2}\right) \equiv m\left(\frac{f^{(k+1)} - a'_1}{f^{(k)} - a_1} - \frac{f^{(k+1)} - a'_2}{f^{(k)} - a_2}\right).$$

On integration, we have

$$\left(\frac{f - a_1}{f - a_2}\right)^n \equiv c_1 \left(\frac{f^{(k)} - a_1}{f^{(k)} - a_2}\right)^m, \quad (2.23)$$

where $c_1 \in \mathbb{C} \setminus \{0\}$. First we suppose that $n \neq m$. Then using Lemma 2.4, we get from (2.23) that $nT(r, f) = mT(r, f^{(k)}) + S(r, f)$, which contradicts the fact that $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Next we suppose that $n = m$. Then from (2.23), we have

$$\frac{f - a_1}{f - a_2} \equiv c_2 \frac{f^{(k)} - a_1}{f^{(k)} - a_2}, \quad (2.24)$$

where $c_2 \in \mathbb{C} \setminus \{0\}$. If $c_2 = 1$, then from (2.24), we have $f \equiv f^{(k)}$, which is a contradiction. Hence $c_2 \neq 1$. Now from (2.24), we get

$$\frac{1 - c_2}{c_2} \frac{f - a_4}{f - a_2} \equiv \frac{a_2 - a_1}{f^{(k)} - a_2}, \quad (2.25)$$

where $a_4 = \frac{a_1 - a_2 c_2}{1 - c_2}$ such that $a_4 \neq a_1, a_2$. Since $f \in \mathcal{E}$ and $f, f^{(k)}$ share a_2 IM, from (2.25), we conclude that $N(r, a_4; f) = S(r, f)$. Also we see that

$$\frac{f - a_1}{a_2 - a_1} + \frac{c_2}{1 - c_2} = \frac{f - a_4}{a_2 - a_1}.$$

Now using the second fundamental theorem, we get from (2.14) and (2.18) that

$$\begin{aligned}
2T(r, f) &= 2T(r, F) + S(r, f) \\
&\leq \bar{N}(r, 0; F) + \bar{N}(r, 1; F) + \bar{N}\left(r, -\frac{c_2}{1 - c_2}; F\right) + S(r, f) \\
&= \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f) \\
&= T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction.

Case 2. Suppose that $H_{nm} \not\equiv 0$ for all $m, n \in \mathbb{N}$. Let $z_{m,n} \in S_{(m,n)}(a_1) \cup S_{(m,n)}(a_2)$ such that $a_1(z_{m,n}), a_2(z_{m,n}) \neq 0, \infty$ and $a_1(z_{m,n}) - a_2(z_{m,n}) \neq 0$. Now from (2.1) and (2.2), we see that

$$H_{nm} = (f - f^{(k)}) \left[\left(n \frac{f' - a_2'}{f - a_2} - m \frac{f^{(k+1)} - a_2'}{f^{(k)} - a_2} \right) - \left(n \frac{f' - a_1'}{f - a_1} - m \frac{f^{(k+1)} - a_1'}{f^{(k)} - a_1} \right) \right],$$

and so $H_{nm}(z_{m,n}) = 0$. Therefore using the first fundamental theorem, we also have,

$$\begin{aligned} & \bar{N}_{(m,n)}(r, a_1; f) + \bar{N}_{(m,n)}(r, a_2; f) \\ & \leq N(r, 0; H_{nm}) + \sum_{i=1}^2 N(r, 0; a_i) + \sum_{i=1}^2 N(r, \infty; a_i) \\ & \quad + N(r, 0; a_1 - a_2) + S(r, f) \\ & \leq T(r, H_{nm}) + S(r, f) \leq T(r, \phi) + T(r, \psi) + S(r, f) = S(r, f). \end{aligned} \quad (2.26)$$

Finally from (2.14) and (2.26), we have

$$\begin{aligned} T(r, f) &= \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f) \\ &= \sum_{m,n} (\bar{N}_{(m,n)}(r, a_1; f) + \bar{N}_{(m,n)}(r, a_2; f)) + S(r, f) \\ &= \sum_{m+n \geq 5} (\bar{N}_{(m,n)}(r, a_1; f) + \bar{N}_{(m,n)}(r, a_2; f)) + S(r, f) \\ &\leq \frac{1}{5} (N(r, a_1; f) + N(r, a_1; f^{(k)}) + N(r, a_2; f) + N(r, a_2; f^{(k)})) + S(r, f) \\ &\leq \frac{1}{5} (2T(r, f) + 2T(r, f^{(k)})) + S(r, f) = \frac{4}{5} T(r, f) + S(r, f), \end{aligned}$$

which is impossible here. Hence $f \equiv f^{(k)}$.

3. Proof of the theorem

PROOF (Proof of Theorem 1.3). If possible suppose that f is a non-constant polynomial. Since a small function of a polynomial is a constant, it follows that $a_1, a_2 \in \mathbb{C}$. This contradicts the fact that $a_1' - a_2' \notin \mathbb{C}$. Hence $f \in \mathcal{E}_T(\mathbb{C})$. Now we divide the following two cases.

Case 1. Suppose that $\phi \not\equiv 0$, where ϕ is defined by (2.1). Clearly $f \not\equiv f^{(k)}$. Now from the proof of Lemma 2.7, we see that $\phi \in \mathcal{S}(f)$. Also from (2.17), we have

$$\begin{aligned} & \bar{N}_{(2)}(r, 0; f - f^{(k)} \mid f = a_1, a_2) + N(r, 0; \Delta(f)(f - f^{(k)}) \mid f \neq a_1, a_2) \\ &= S(r, f). \end{aligned} \quad (3.1)$$

On the other hand, from (2.15) and (2.16), we have

$$\bar{N}(r, a_1; f, f^{(k)} \mid \geq 2) + \bar{N}(r, a_2; f, f^{(k)} \mid \geq 2) = S(r, f). \quad (3.2)$$

Let ψ be defined by (2.2). Since $\Delta(f^{(k)}) \neq 0$, it follows that $\psi \neq 0$. Now rewriting (2.1), we get

$$f' = \frac{\alpha_{1,2}f^2 + \alpha_{1,1}f + \alpha_{1,0} + Q_1}{f - f^{(k)}}, \quad (3.3)$$

where $\alpha_{1,2} = \frac{a'_1 - a'_2 - \phi}{a_1 - a_2}$, $\alpha_{1,1} = a'_1 - a_1 \frac{a'_1 - a'_2}{a_1 - a_2} + \frac{(a_1 + a_2)\phi}{a_1 - a_2}$, $\alpha_{1,0} = -\frac{\phi a_1 a_2}{a_1 - a_2}$ and $Q_1 = -\frac{a'_1 - a'_2}{a_1 - a_2} f f^{(k)} - \left(a'_1 - a_1 \frac{a'_1 - a'_2}{a_1 - a_2} \right) f^{(k)}$. Now we divide the following two sub-cases.

Sub-case 1.1. Suppose that $\phi \neq a'_1 - a'_2$. Certainly $\alpha_{1,2} \neq 0$. Now by induction and using (3.3) repeatedly, we obtain the following

$$f^{(k)} = \frac{\sum_{j=0}^{2k} \alpha_{k,j} f^j + Q_k}{(f - f^{(k)})^{2k-1}}, \quad (3.4)$$

where

$$Q_k = \sum_{\substack{l < 2k \\ l+j_1+j_2+\dots+j_k \leq 2k}} \beta_{l,j_1,j_2,\dots,j_k} f^l (f^{(k)})^{j_1} (f^{(k+1)})^{j_2} \dots (f^{(2k-1)})^{j_k}. \quad (3.5)$$

Here $\alpha_{k,j}, \beta_{l,j_1,j_2,\dots,j_k} \in \mathcal{S}(f)$ and $\psi_i := \alpha_{i,2i}$ satisfies the recurrence formula

$$\psi_1 = \alpha_{1,2}, \quad \psi_{i+1} = \psi'_i + \psi_1 \psi_i, \quad i = 1, 2, \dots, k-1. \quad (3.6)$$

From the recurrence formula (3.6) for ψ_i , we can easily derive the expression

$$\psi_k = \psi_1^k + Q(\psi_1), \quad (3.7)$$

where $Q(\psi_1)$ is a differential polynomial in ψ_1 with a degree less than or equal to $k-1$.

Now we divide the following two sub-cases.

Sub-case 1.1.1. Suppose that $\psi_k = \alpha_{k,2k} \neq 0$. Then using (2.19) and the lemma of logarithmic derivative, we get from (3.5) that

$$m\left(r, \frac{Q_k}{f^{2k-1} f^{(k)}}\right) = S(r, f). \quad (3.8)$$

Again from (3.4), we have

$$\sum_{j=0}^{2k} \alpha_{k,j} f^j = f^{(k)}(f - f^{(k)})^{2k-1} - Q_k. \quad (3.9)$$

Now from Lemma 2.4, (3.8) and (3.9), we obtain that

$$\begin{aligned} 2kT(r, f) + S(r, f) &= T\left(r, \sum_{j=0}^{2k} \alpha_{k,j} f^j\right) \\ &\leq m\left(r, \left(1 - \frac{f^{(k)}}{f}\right)^{2k-1} - \frac{Q_k}{f^{2k-1}f^{(k)}}\right) \\ &\quad + m(r, f^{2k-1}f^{(k)}) + S(r, f) \\ &\leq m\left(r, \left(1 - \frac{f^{(k)}}{f}\right)^{2k-1}\right) + m\left(r, \frac{Q_k}{f^{2k-1}f^{(k)}}\right) \\ &\quad + m(r, f^{2k-1}) + m(r, f^{(k)}) + S(r, f) \\ &\leq (2k-1)T(r, f) + T(r, f^{(k)}) + S(r, f), \end{aligned}$$

i.e., $T(r, f) \leq T(r, f^{(k)}) + S(r, f)$. Since $f \in \mathcal{E}_T(\mathbb{C})$, we have $T(r, f^{(k)}) \leq T(r, f) + S(r, f)$. Therefore $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Consequently, from Lemma 2.7, one can see that $f \equiv f^{(k)}$, which is impossible here.

Sub-case 1.1.2. Suppose that $\psi_k = \alpha_{k,2k} \equiv 0$. Then from (3.6), we have $\psi'_{k-1} + \psi_1 \psi_{k-1} \equiv 0$. On integration, we have $\psi_{k-1}(z) = c_0 e^{\xi(z)}$, where $\xi(z) = -\int_0^z \psi_1(z) dz \in \mathcal{M}(\mathbb{C})$ and $c_0 \in \mathbb{C} \setminus \{0\}$. We know that if $\xi(z)$ has a pole at the point z_0 , then z_0 is an essential singularity of $e^{\xi(z)}$. Since $\psi_{k-1} \in \mathcal{M}(\mathbb{C})$, it follows that $\xi \in \mathcal{E}(\mathbb{C})$ and so $\psi_1 \in \mathcal{E}(\mathbb{C})$. On the other hand, we see that if ψ_1 is a polynomial, then ψ_{i+1} is also a polynomial for $i = 1, 2, \dots, k-1$. In that case we arrive at a contradiction. Hence $\psi_1 \in \mathcal{E}_T(\mathbb{C})$. Now from (3.7), we see that $\psi_1^k = -Q(\psi_1)$. Using Lemma 2.1, we evaluate that $m(r, \psi_1) = S(r, \psi_1)$. Since $N(r, \psi_1) = 0$, it follows that $T(r, \psi_1) = S(r, \psi_1)$, which is impossible.

Sub-case 1.2. Suppose that $\phi = a'_1 - a'_2$. Now from (2.1), we have

$$\begin{aligned} f - f^{(k)} &= \frac{(a'_1 - a'_2)(f - a_1)(f - a_2)}{(a'_1 - a'_2)(f - a_1) - (a_1 - a_2)(f' - a'_1)} \\ &= \frac{(a'_1 - a'_2)(f - a_1)(f - a_2)}{\Delta(f)}. \end{aligned} \quad (3.10)$$

Since $(a'_1 - a'_2)(f - a_1) - (a_1 - a_2)(f' - a'_1) = (a'_1 - a'_2)(f - a_2) - (a_1 - a_2) \cdot (f' - a'_2)$, from (3.10), we have

$$\begin{aligned} f^{(k)} - a_1 &= -\frac{(a_1 - a_2)(f - a_1)(f' - a'_2)}{\Delta(f)} \quad \text{and} \\ f^{(k)} - a_2 &= -\frac{(a_1 - a_2)(f - a_2)(f' - a'_1)}{\Delta(f)}. \end{aligned} \quad (3.11)$$

Also from (2.5), we have

$$\begin{aligned} &(a''_1 - a''_2)(f - a_1)^2 - (a''_1 - a''_2)(f - a_1)(f^{(k)} - a_1) \\ &\quad - (a_1 - a_2)(f - a_1)(f'' - a''_1) + (a_1 - a_2)(f'' - a''_1)(f^{(k)} - a_1) \\ &\quad + (a'_1 - a'_2)(f - a_1)(f' - a'_1) - (a'_1 - a'_2)(f - a_1)(f^{(k+1)} - a'_1) \\ &\quad - (a_1 - a_2)(f' - a'_1)^2 + (a_1 - a_2)(f' - a'_1)(f^{(k+1)} - a'_1) \\ &= (a''_1 - a''_2)(f - a_1)(f - a_2) + (a'_1 - a'_2)(f' - a'_1)(f - a_2) \\ &\quad + (a'_1 - a'_2)(f - a_1)(f' - a'_2). \end{aligned} \quad (3.12)$$

Let $z_{p,1} \in S_{(p,1)}(a_1)$ ($p \geq 2$) such that $a_1(z_{p,1}) - a_2(z_{p,1}) \neq 0, \infty$ and $a'_1(z_{p,1}) - a'_2(z_{p,1}) \neq 0$. Clearly, from (2.10), we have

$$f^{(k+1)}(z_{p,1}) - a'_1(z_{p,1}) = c_1 = \frac{a'_1(z_{p,1}) - a'_2(z_{p,1})}{p}.$$

In some neighbourhood of $z_{p,1}$, it is easy to calculate, from (2.7), (2.8) and (3.12) that

$$\begin{aligned} &(b_p c_2 p^2 (p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) - b_p(a'_1(z_{p,1}) - a'_2(z_{p,1}))^2 \\ &\quad + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a'_1(z_{p,1}) - a'_2(z_{p,1}))) \\ &\quad - p b_p((a''_1(z_{p,1}) - a''_2(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a'_1(z_{p,1}) - a'_2(z_{p,1}))^2) \\ &\quad - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a'_1(z_{p,1}) - a'_2(z_{p,1}))) (z - z_{p,1})^p \\ &\quad + A_{p+1}(z - z_{p,1})^{p+1} + \dots \equiv 0 \quad (A_{p+1} \in \mathbb{C}), \end{aligned}$$

which shows that

$$\begin{aligned} &(b_p c_2 p^2 (p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) - b_p(a'_1(z_{p,1}) - a'_2(z_{p,1}))^2 \\ &\quad + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a'_1(z_{p,1}) - a'_2(z_{p,1}))) \end{aligned}$$

$$\begin{aligned}
& -pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2) \\
& - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) = 0. \tag{3.13}
\end{aligned}$$

On the other hand, from (2.6), we have

$$\begin{aligned}
& (a_1''' - a_2''')(f - a_1)^2 - (a_1''' - a_2''')(f - a_1)(f^{(k)} - a_1) \\
& + (a_1'' - a_2'')(f - a_1)(f' - a_1') - (a_1'' - a_2'')(f' - a_1')(f^{(k)} - a_1) \\
& - (a_1' - a_2')(f - a_1)(f'' - a_1'') + (a_1' - a_2')(f'' - a_1'')(f^{(k)} - a_1) \\
& - (a_1 - a_2)(f - a_1)(f''' - a_1''') + (a_1 - a_2)(f''' - a_1''')(f^{(k)} - a_1) \\
& + 2(a_1'' - a_2'')(f - a_1)(f' - a_1') - 2(a_1'' - a_2'')(f - a_1)(f^{(k+1)} - a_1') \\
& - 2(a_1 - a_2)(f' - a_1')(f'' - a_1'') + (a_1' - a_2')(f - a_1)(f'' - a_1'') \\
& + 2(a_1 - a_2)(f'' - a_1'')(f^{(k+1)} - a_1') - (a_1' - a_2')(f - a_1)(f^{(k+2)} - a_1'') \\
& - (a_1 - a_2)(f' - a_1')(f'' - a_1'') + (a_1 - a_2)(f' - a_1')(f^{(k+2)} - a_1'') \\
& = (a_1''' - a_2''')(f - a_1)(f - a_2) + 2(a_1'' - a_2'')(f' - a_1')(f - a_2) \\
& + 2(a_1'' - a_2'')(f - a_1)(f' - a_2') + (a_1' - a_2')(f'' - a_1'')(f - a_2) \\
& + 2(a_1' - a_2')(f' - a_1')(f' - a_2') + (a_1' - a_2')(f - a_1)(f'' - a_1''). \tag{3.14}
\end{aligned}$$

In some neighbourhood of $z_{p,1}$, it is easy to calculate, from (2.7), (2.8) and (3.14) that

$$\begin{aligned}
& (b_p c_2 p^2 (p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) + b_p(p-1)(a_1'(z_{p,1}) - a_2'(z_{p,1}))^2 \\
& + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) \\
& - 2pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2) \\
& - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1}))(z - z_{p,1})^{p-1} \\
& + B_p(z - z_{p,1})^p + \cdots \equiv 0 \quad (B_p \in \mathbb{C}),
\end{aligned}$$

which shows that

$$\begin{aligned}
& b_p c_2 p^2 (p+1)(a_1(z_{p,1}) - a_2(z_{p,1})) + b_p(p-1)(a_1'(z_{p,1}) - a_2'(z_{p,1}))^2 \\
& + b_{p+1}(p+1)^2(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) \\
& - 2pb_p((a_1''(z_{p,1}) - a_2''(z_{p,1}))(a_1(z_{p,1}) - a_2(z_{p,1})) + (a_1'(z_{p,1}) - a_2'(z_{p,1}))^2) \\
& - b_{p+1}p(p+1)(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1'(z_{p,1}) - a_2'(z_{p,1})) = 0. \tag{3.15}
\end{aligned}$$

Now from (3.13) and (3.15), we have

$$b_p p(a_1(z_{p,1}) - a_2(z_{p,1}))(a_1''(z_{p,1}) - a_2''(z_{p,1})) = 0. \quad (3.16)$$

Since $b_p \neq 0$ and $a_1(z_{p,1}) - a_2(z_{p,1}) \neq 0, \infty$, from (3.16) we have $a_1''(z_{p,1}) - a_2''(z_{p,1}) = 0$.

Now since $a_1' - a_2' \notin \mathbb{C}$, it follows that

$$\sum_{p \geq 2} \bar{N}_{(p,1)}(r, a_1; f) \leq N(r, 0; a_1'' - a_2'') \leq S(r, f). \quad (3.17)$$

Let $\hat{z}_{p,1} \in S_{(p,1)}(a_2)$ ($p \geq 2$) such that $a_1(\hat{z}_{p,1}) - a_2(\hat{z}_{p,1}) \neq 0, \infty$ and $a_1'(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1}) \neq 0$. Clearly from (2.12), we have

$$f^{(k+1)}(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1}) = \hat{c}_1 = -\frac{a_1'(\hat{z}_{p,1}) - a_2'(\hat{z}_{p,1})}{p}.$$

Now proceeding in the same way as done above and using (2.11) instead of (2.7), one can easily deduce that $a_1''(\hat{z}_{p,1}) - a_2''(\hat{z}_{p,1}) = 0$ and so

$$\sum_{p \geq 2} \bar{N}_{(p,1)}(r, a_2; f) \leq N(r, 0; a_1'' - a_2'') \leq S(r, f). \quad (3.18)$$

Therefore from (3.2), (3.17) and (3.18), we see that

$$\begin{aligned} \bar{N}_{(2)}(r, a_1; f) + \bar{N}_{(2)}(r, a_2; f) &\leq \sum_{p \geq 2} (\bar{N}_{(p,1)}(r, a_1; f) + \bar{N}_{(p,1)}(r, a_2; f)) \\ &= S(r, f). \end{aligned} \quad (3.19)$$

Let $l \in \mathbb{N}$ and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{(l+1)}(r, a; f)$ to denote the counting function of a -points of f with multiplicity greater than l . Similarly, $\bar{N}_{(l+1)}(r, a; f)$ is its reduced function. Now we divide the following three sub-cases.

Sub-case 1.2.1. Suppose that

$$\bar{N}_{(2)}(r, a_1; f^{(k)}) = S(r, f) \quad \text{and} \quad \bar{N}_{(2)}(r, a_2; f^{(k)}) = S(r, f).$$

Then from (3.2), one can easily obtain that

$$\sum_{q \geq 2} (\bar{N}_{(1,q)}(r, a_1; f) + \bar{N}_{(1,q)}(r, a_2; f)) = S(r, f). \quad (3.20)$$

Now from (2.14), (3.19) and (3.20), we deduce that

$$\begin{aligned}
T(r, f) &= \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f) \\
&= \sum_{i=1}^2 \bar{N}_{(1,1)}(r, a_i; f) + \sum_{p \geq 2} (\bar{N}_{(p,1)}(r, a_1; f) + \bar{N}_{(p,1)}(r, a_2; f)) \\
&\quad + \sum_{q \geq 2} (\bar{N}_{(1,q)}(r, a_1; f) + \bar{N}_{(1,q)}(r, a_2; f)) + S(r, f) \\
&= \bar{N}_{(1,1)}(r, a_1; f) + \bar{N}_{(1,1)}(r, a_2; f) + S(r, f). \tag{3.21}
\end{aligned}$$

Let $\alpha \in \mathcal{S}(f)$ be arbitrary. Now using the first fundamental theorem, we get from (2.2), Lemmas 2.2 and 2.3 that

$$\begin{aligned}
m(r, \psi) &= m\left(r, \frac{\Delta(f^{(k)})(f^{(k)} - \alpha)}{(f^{(k)} - a_1)(f^{(k)} - a_2)} \left(\frac{f - \alpha}{f^{(k)} - \alpha} - 1\right)\right) \\
&\leq m\left(r, \frac{\Delta(f^{(k)})(f^{(k)} - \alpha)}{(f^{(k)} - a_1)(f^{(k)} - a_2)}\right) + m\left(r, \frac{f - \alpha}{f^{(k)} - \alpha} - 1\right) \\
&\leq m\left(r, \frac{f - \alpha}{f^{(k)} - \alpha}\right) + S(r, f) \\
&= T\left(r, \frac{f^{(k)} - \alpha}{f - \alpha}\right) - N\left(r, \frac{f - \alpha}{f^{(k)} - \alpha}\right) + S(r, f) \\
&= m\left(r, \frac{f^{(k)} - \alpha}{f - \alpha}\right) + N\left(r, \frac{f^{(k)} - \alpha}{f - \alpha}\right) - N\left(r, \frac{f - \alpha}{f^{(k)} - \alpha}\right) + S(r, f) \\
&\leq m\left(r, \frac{f^{(k)} - \alpha^{(k)}}{f - \alpha}\right) + m\left(r, \frac{\alpha^{(k)} - \alpha}{f - \alpha}\right) \\
&\quad + N(r, \alpha; f) - N(r, \alpha; f^{(k)}) + S(r, f) \\
&\leq m(r, \alpha; f) + N(r, \alpha; f) - N(r, \alpha; f^{(k)}) + S(r, f) \\
&= T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f).
\end{aligned}$$

Under the given conditions, we have $N(r, \psi) = S(r, f)$. Consequently we have

$$T(r, \psi) \leq T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f). \tag{3.22}$$

Now we consider the following two sub-cases.

Sub-case 1.2.1.1. Suppose that $H_{11} \equiv 0$. Then immediately we have $T(r, f) = T(r, f^{(k)}) + S(r, f)$ and so by Lemma 2.5, we get $f \equiv f^{(k)}$, which is impossible.

Sub-case 1.2.1.2. Suppose that $H_{11} \neq 0$. Let $z_{1,1} \in S_{(1,1)}(a_1) \cup S_{(1,1)}(a_2)$. Then it is easy to obtain that $H_{11}(z_{1,1}) = 0$ and so we conclude that

$$\begin{aligned} \sum_{i=1}^2 \bar{N}_{(1,1)}(r, a_i; f) &\leq N(r, 0; H_{11}) + S(r, f) \\ &\leq T(r, H_{11}) + S(r, f) \leq T(r, \psi) + S(r, f). \end{aligned} \quad (3.23)$$

Then from (3.21), (3.22) and (3.23), we have

$$\begin{aligned} T(r, f) &\leq T(r, \psi) + S(r, f) \leq T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f), \\ \text{i.e., } N(r, \alpha; f^{(k)}) &= S(r, f), \end{aligned}$$

where $\alpha \in \mathcal{S}(f)$ is arbitrary. In particular we have $\bar{N}(r, a_1; f^{(k)}) + \bar{N}(r, a_2; f^{(k)}) = S(r, f)$. Since f and $f^{(k)}$ share a_1 and a_2 IM, we have $\bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) = S(r, f)$ and so $\bar{N}_{(1,1)}(r, a_1; f) + \bar{N}_{(1,1)}(r, a_2; f) = S(r, f)$. Therefore from (3.21), we arrive at a contradiction.

Sub-case 1.2.2. Suppose that

$$\begin{aligned} &\text{either } \bar{N}_{(2)}(r, a_1; f^{(k)}) = S(r, f) \text{ and } \bar{N}_{(2)}(r, a_2; f^{(k)}) \neq S(r, f) \\ &\text{or } \bar{N}_{(2)}(r, a_1; f^{(k)}) \neq S(r, f) \text{ and } \bar{N}_{(2)}(r, a_2; f^{(k)}) = S(r, f). \end{aligned}$$

Without loss of generality we may assume that $\bar{N}_{(2)}(r, a_1; f^{(k)}) = S(r, f)$ and $\bar{N}_{(2)}(r, a_2; f^{(k)}) \neq S(r, f)$.

Now from (3.2), one can easily show that

$$\sum_{q \geq 2} \bar{N}_{(1,q)}(r, a_1; f) = S(r, f). \quad (3.24)$$

Consequently from (3.19) and (3.24), we deduce that

$$\begin{aligned} \bar{N}(r, a_1; f) &= \sum_{p,q} \bar{N}_{(p,q)}(r, a_1; f) \\ &= \bar{N}_{(1,1)}(r, a_1; f) + \sum_{p \geq 2} \bar{N}_{(p,1)}(r, a_1; f) + \sum_{q \geq 2} \bar{N}_{(1,q)}(r, a_1; f) \\ &= \bar{N}_{(1,1)}(r, a_1; f) + S(r, f). \end{aligned} \quad (3.25)$$

Let

$$\varphi_1 = \frac{f - f^{(k)}}{(f - a_1)(f - a_2)}. \quad (3.26)$$

Clearly $\varphi_1 \neq 0$. Since $\phi = a'_1 - a'_2$, from (2.1) and (3.26), we have

$$(a'_1 - a'_2) \left(f - a_1 - \frac{1}{\varphi_1} \right) = (a_1 - a_2)(f' - a'_1). \quad (3.27)$$

Let $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ ($q \geq 2$) such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$, $a'_1(\hat{z}_{1,q}) - a'_2(\hat{z}_{1,q}) \neq 0$ and $\varphi_1(\hat{z}_{1,q}) \neq 0, \infty$. Clearly $\Delta(f(\hat{z}_{1,q})) \neq 0$ and so from (3.11), we conclude that $\hat{z}_{1,q}$ is a zero of $f' - a'_1$ of multiplicity $q - 1$. On the other hand, from (3.27), we conclude that $\hat{z}_{1,q}$ is a zero of $f - a_1 - \frac{1}{\varphi_1}$, i.e., $f(\hat{z}_{1,q}) = a_1(\hat{z}_{1,q}) + \frac{1}{\varphi_1(\hat{z}_{1,q})}$. Also since f and $f^{(k)}$ share a_2 IM, it follows that $f(\hat{z}_{1,q}) = a_2(\hat{z}_{1,q})$. Consequently we have $\varphi_1(\hat{z}_{1,q}) = \frac{1}{a_2(\hat{z}_{1,q}) - a_1(\hat{z}_{1,q})}$.

We claim that $\varphi_1 \notin \mathbb{C}$. If possible suppose that $\varphi_1 \in \mathbb{C} \setminus \{0\}$. Then we have $\varphi_1 = \frac{1}{a_2(\hat{z}_{1,q}) - a_1(\hat{z}_{1,q})}$. If $\varphi_1 \neq \frac{1}{a_2 - a_1}$, then

$$\sum_{q \geq 2} \bar{N}_{(1,q)}(r, a_2; f) \leq N \left(r, 0; \varphi_1 - \frac{1}{a_2 - a_1} \right) \leq S(r, f). \quad (3.28)$$

Also from (3.2), we have $\bar{N}(r, a_2; f, f^{(k)} | \geq 2) = S(r, f)$ and so from (3.28), we conclude that $\bar{N}_{(2)}(r, a_2; f^{(k)}) = S(r, f)$, which is impossible. Hence $\varphi_1 \equiv \frac{1}{a_2 - a_1}$. This shows that $a_2 - a_1 \in \mathbb{C}$, which is again impossible. Hence $\varphi_1 \notin \mathbb{C}$. Let $\phi_1 = \frac{\varphi'_1}{\varphi_1}$. Clearly $\phi_1 \neq 0$.

Let $z_{1,q} \in S_{(1,q)}(a_1)$ such that $a_1(z_{1,q}) - a_2(z_{1,q}) \neq 0, \infty$. Then $z_{1,q}$ is a zero of $f - f^{(k)}$. Consequently $\varphi_1(z_{1,q}) \neq \infty$. Similarly if $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$, then $\varphi_1(\hat{z}_{1,q}) \neq \infty$. Now from (3.1), (3.2) and (3.19), we conclude that $\bar{N}(r, 0; \varphi_1) + N(r, \varphi_1) = S(r, f)$. Then $\phi_1 \in \mathcal{S}(f)$. Now by logarithmic differentiation, we get from (3.26) that

$$\phi_1 = \frac{f' - f^{(k+1)}}{f - f^{(k)}} - \frac{f' - a'_1}{f - a_1} - \frac{f' - a'_2}{f - a_2}, \quad \text{i.e.,}$$

$$\begin{aligned} & \phi_1(f - a_1)^2(f - a_2) - \phi_1(f - a_1)(f - a_2)(f^{(k)} - a_1) \\ &= -(f - a_1)(f - a_2)(f^{(k+1)} - a'_1) + (f' - a'_1)(f - a_2)(f^{(k)} - a_1) \\ &+ (f - a_1)(f' - a'_2)(f^{(k)} - a_1) - (f - a_1)^2(f' - a'_2) \end{aligned} \quad (3.29)$$

or

$$\begin{aligned} & \phi_1(f - a_1)(f - a_2)^2 - \phi_1(f - a_1)(f - a_2)(f^{(k)} - a_2) \\ &= -(f - a_1)(f - a_2)(f^{(k+1)} - a'_2) + (f' - a'_1)(f - a_2)(f^{(k)} - a_2) \\ &+ (f - a_1)(f' - a'_2)(f^{(k)} - a_2) - (f' - a'_1)(f - a_2)^2. \end{aligned} \quad (3.30)$$

Let $\hat{z}_{1,q} \in S_{(1,q)}(a_2)$ ($q \geq 3$) such that $a_1(\hat{z}_{1,q}) - a_2(\hat{z}_{1,q}) \neq 0, \infty$ and $a'_1(\hat{z}_{1,q}) - a'_2(\hat{z}_{1,q}) \neq 0$. Clearly $\Delta(f(\hat{z}_{1,q})) \neq 0$ and so from (3.11), we conclude that $\hat{z}_{1,q}$ is a zero of $f' - a'_1$ of multiplicity $q - 1$. Then from (3.30), it is easy to calculate $\phi_1(\hat{z}_{1,q}) = 0$. Since $\phi_1 \not\equiv 0$, it follows that

$$\sum_{q \geq 3} \bar{N}_{(1,q)}(r, a_2; f) \leq N(r, 0; \phi_1) \leq S(r, f). \quad (3.31)$$

Now from (3.2) and (3.31), we can easily conclude that

$$\bar{N}_{(3)}(r, a_2; f^{(k)}) = S(r, f). \quad (3.32)$$

Now we consider the following two sub-cases.

Sub-case 1.2.2.1. Suppose that $H \equiv 0$. Then on integration, we have

$$\frac{f - a_1}{f - a_2} \equiv c_2 \frac{f^{(k)} - a_1}{f^{(k)} - a_2},$$

where $c_2 \in \mathbb{C} \setminus \{0\}$. Now by Lemma 2.4, we conclude that $T(r, f) = T(r, f^{(k)}) + S(r, f)$ and so by Lemma 2.5, we have $f \equiv f^{(k)}$, which is impossible.

Sub-case 1.2.2.2. Suppose that $H \not\equiv 0$. It is easy to obtain that $m(r, H) = S(r, f)$ and $N(r, H) = \bar{N}_{(1,2)}(r, a_2; f)$. Therefore

$$T(r, H) = m(r, H) + N(r, H) = \bar{N}_{(1,2)}(r, a_2; f) + S(r, f). \quad (3.33)$$

Now from (2.13) and (2.14), we have $T(r, f) = m(r, f - f^{(k)}) + S(r, f)$. Therefore from (2.1), (2.2), (3.22) and (3.33), we have

$$\begin{aligned} T(r, f) &= m\left(r, \frac{H(f - f^{(k)})}{H}\right) + S(r, f) \\ &= m\left(r, \frac{\phi - \psi}{H}\right) + S(r, f) \\ &= T\left(r, \frac{H}{\phi - \psi}\right) + S(r, f) \\ &\leq T(r, \psi) + T(r, H) + S(r, f) \\ &\leq T(r, f) + \bar{N}_{(1,2)}(r, a_2; f) - N(r, \alpha; f^{(k)}) + S(r, f) \end{aligned}$$

and so

$$N(r, \alpha; f^{(k)}) \leq \bar{N}_{(1,2)}(r, a_2; f) + S(r, f). \quad (3.34)$$

Suppose that $\alpha = a_2$. Then from (3.34), we have

$$\bar{N}(r, a_2; f^{(k)}) \leq \bar{N}_{(1,2)}(r, a_2; f) + S(r, f). \quad (3.35)$$

Since f and $f^{(k)}$ share a_2 IM, from (3.2) and (3.19), it follows that

$$\bar{N}(r, a_2; f^{(k)}) + S(r, f) = \bar{N}(r, a_2; f) + S(r, f) = \sum_{i=1}^2 \bar{N}_{(1,i)}(r, a_2; f) + S(r, f),$$

and so from (3.35), we conclude that $\bar{N}_{(1,1)}(r, a_2; f) = S(r, f)$. Let

$$G_1 = \frac{f^{(k+1)} - a'_1}{f^{(k)} - a_1} - \frac{f' - a'_1}{f - a_1} - \frac{a'_1 - a'_2}{a_1 - a_2}. \quad (3.36)$$

If $G_1 \equiv 0$, then on integration we have $f^{(k)} - a_1 = c_3(a_1 - a_2)(f - a_1)$, where $c_3 \in \mathbb{C} \setminus \{0\}$, and so by Lemma 2.4, we get $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Now by Lemma 2.7, we conclude that $f \equiv f^{(k)}$, which is impossible here. Hence $G_1 \not\equiv 0$. Also from (3.25), it is easy to prove that $N(r, G_1) = S(r, f)$. Since $m(r, G_1) = S(r, f)$, it follows that $G_1 \in \mathcal{S}(f)$.

Let $\hat{z}_{1,2} \in S_{(1,2)}(a_2)$ such that $a_1(\hat{z}_{1,2}) - a_2(\hat{z}_{1,2}) \neq 0, \infty$, $a'_1(\hat{z}_{1,2}) - a'_2(\hat{z}_{1,2}) \neq 0$ and $\phi(\hat{z}_{1,2}) \neq 0, \infty$. Then $f^{(k)}(\hat{z}_{1,2}) = a_2(\hat{z}_{1,2})$ and $f^{(k+1)}(\hat{z}_{1,2}) = a'_2(\hat{z}_{1,2})$. Also from (3.11), one can easily conclude that $\hat{z}_{1,2}$ is a simple zero of $f' - a'_1$, i.e., $f'(\hat{z}_{1,2}) = a'_1(\hat{z}_{1,2})$.

Now from (3.36), we conclude that $G_1(\hat{z}_{1,2}) = 0$ and so $\bar{N}_{(1,2)}(r, a_2; f) \leq N(r, 0; G_1) + S(r, f) \leq S(r, f)$. Consequently, $\bar{N}_{(2)}(r, a_2; f) = S(r, f)$, which is impossible.

Sub-case 1.2.3. Suppose that

$$\bar{N}_{(2)}(r, a_1; f^{(k)}) \neq S(r, f) \quad \text{and} \quad \bar{N}_{(2)}(r, a_2; f^{(k)}) \neq S(r, f).$$

Let $z_{1,q} \in S_{(1,q)}(a_1)$ ($q \geq 3$) such that $a_1(z_{1,q}) - a_2(z_{1,q}) \neq 0, \infty$ and $a'_1(z_{1,q}) - a'_2(z_{1,q}) \neq 0$. Clearly, $\Delta(f(z_{1,q})) \neq 0$ and so from (3.11), we conclude that $z_{1,q}$ is a zero of $f' - a'_2$ of multiplicity $q - 1$. Now from (3.29), we calculate that $\phi_1(z_{1,q}) = 0$. Since $\phi_1 \not\equiv 0$, we get

$$\sum_{q \geq 3} \bar{N}_{(1,q)}(r, a_1; f) \leq N(r, 0; \phi_1) \leq S(r, f). \quad (3.37)$$

Now, from (3.2) and (3.37), we can easily conclude that

$$\bar{N}_{(3)}(r, a_1; f^{(k)}) = S(r, f). \quad (3.38)$$

Also, from (3.32), we have

$$\bar{N}_{(3)}(r, a_2; f^{(k)}) = S(r, f). \quad (3.39)$$

In this case from, (2.14), we have

$$T(r, f) = \sum_{i=1}^2 (\bar{N}_{(1,1)}(r, a_i; f) + \bar{N}_{(1,2)}(r, a_i; f)) + S(r, f). \quad (3.40)$$

From Sub-case 1.2.1.1, we conclude that $H_{11} \neq 0$. Let $z_{1,1} \in S_{(1,1)}(a_1) \cup S_{(1,1)}(a_2)$. Then from (3.22) and (3.23), we see that

$$\sum_{i=1}^2 \bar{N}_{(1,1)}(r, a_i; f) \leq T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f),$$

where $\alpha \in \mathcal{S}(f)$ is arbitrary, and so from (3.40), we have

$$\begin{aligned} N(r, \alpha; f^{(k)}) &\leq \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) + S(r, f), \\ \text{i.e., } \bar{N}(r, \alpha; f^{(k)}) &\leq \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) + S(r, f). \end{aligned} \quad (3.41)$$

Suppose $\alpha = a_1$. Since

$$\bar{N}(r, a_1; f^{(k)}) = \bar{N}(r, a_1; f) = \bar{N}_{(1,1)}(r, a_1; f) + \bar{N}_{(1,2)}(r, a_1; f) + S(r, f),$$

from (3.41), we conclude that $\bar{N}_{(1,1)}(r, a_1; f) \leq \bar{N}_{(1,2)}(r, a_2; f) + S(r, f)$. Again if we take $\alpha = a_2$, then from (3.41), we can easily deduce that

$$\bar{N}_{(1,1)}(r, a_2; f) \leq \bar{N}_{(1,2)}(r, a_1; f) + S(r, f).$$

Consequently from (3.40), we have

$$T(r, f) \leq 2 \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) + S(r, f). \quad (3.42)$$

Now we divide the following two sub-cases.

Sub-case 1.2.3.1. Suppose that $H_{21} \equiv 0$. Then on integration, we have

$$\left(\frac{f - a_1}{f - a_2} \right)^2 = c_1 \frac{f^{(k)} - a_1}{f^{(k)} - a_2},$$

where $c_1 \in \mathbb{C} \setminus \{0\}$. Now using Lemma 2.4, we deduce that $2T(r, f) = T(r, f^{(k)}) + S(r, f)$. Since $f \in \mathcal{E}_T(\mathbb{C})$, it follows that $T(r, f^{(k)}) \leq T(r, f) + S(r, f)$. Consequently we have $T(r, f) = S(r, f)$, which is impossible.

Sub-case 1.2.3.2. Suppose that $H_{21} \neq 0$. Let $z_{1,2} \in S_{(1,2)}(a_1) \cup S_{(1,2)}(a_2)$. Then it is easy to obtain that $H_{21}(z_{1,2}) = 0$ and so we conclude that

$$\begin{aligned}
\sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) &\leq N(r, 0; H_{21}) + S(r, f) \\
&\leq T(r, H_{21}) + S(r, f) \\
&\leq T(r, \psi) + S(r, f).
\end{aligned} \tag{3.43}$$

Now from (3.22) and (3.43), we see that

$$\sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) \leq T(r, f) - N(r, \alpha; f^{(k)}) + S(r, f), \tag{3.44}$$

where $\alpha \in \mathcal{S}(f)$ is arbitrary. In particular from (3.44), we have

$$\sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) \leq T(r, f) - N(r, a_1; f^{(k)}) + S(r, f) \tag{3.45}$$

$$\text{and} \quad \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) \leq T(r, f) - N(r, a_2; f^{(k)}) + S(r, f). \tag{3.46}$$

Adding (3.45) and (3.46), we have

$$2 \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) \leq 2T(r, f) - N(r, a_1; f^{(k)}) - N(r, a_2; f^{(k)}) + S(r, f). \tag{3.47}$$

Now using (3.42), from (3.47) we get

$$N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) \leq T(r, f) + S(r, f). \tag{3.48}$$

Again, from (3.42) and (3.48), we conclude that

$$N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) \leq 2 \sum_{i=1}^2 \bar{N}_{(1,2)}(r, a_i; f) + S(r, f). \tag{3.49}$$

Note that

$$\sum_{i=1}^2 (\bar{N}_{(1,1)}(r, a_i; f) + 2\bar{N}_{(1,2)}(r, a_i; f)) \leq N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)}) + S(r, f),$$

and so from (3.49), we conclude that

$$\sum_{i=1}^2 \bar{N}_{(1,1)}(r, a_i; f) = S(r, f). \tag{3.50}$$

Now, from (3.40) and (3.50), we deduce that

$$\begin{aligned}
 T(r, f) &= \bar{N}(r, a_1; f) + \bar{N}(r, a_2; f) + S(r, f) \\
 &= \bar{N}_{(1,2)}(r, a_1; f) + \bar{N}_{(1,2)}(r, a_2; f) + S(r, f) \\
 &\leq \bar{N}_{(2)}(r, a_1; f^{(k)}) + \bar{N}_{(2)}(r, a_2; f^{(k)}) + S(r, f) \\
 &\leq \frac{1}{2} (N(r, a_1; f^{(k)}) + N(r, a_2; f^{(k)})) + S(r, f) \\
 &\leq T(r, f^{(k)}) + S(r, f).
 \end{aligned} \tag{3.51}$$

Since $f \in \mathcal{E}_T(\mathbb{C})$, it follows that $T(r, f^{(k)}) \leq T(r, f) + S(r, f)$ and so from (3.51), we conclude that $T(r, f) = T(r, f^{(k)}) + S(r, f)$. Consequently from Lemma 2.7, one can conclude that $f \equiv f^{(k)}$, which is impossible here.

Case 2. Suppose that $\phi \equiv 0$. Since $\Delta(f) \not\equiv 0$, it follows that $f \equiv f^{(k)}$. This completes the proof.

Acknowledgement

The authors wish to thank the referee for his/her valuable comments and suggestions towards the improvement of the paper.

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