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Generalized solution of the double obstacle problem for Musielak-Orlicz Dirichlet energy integral on metric measure spaces

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Abstract. In this paper, we are concerned with the existence and uniqueness of a generalized solution to a double obstacle problem for Musielak-Orlicz Dirichlet energy integral on metric measure spaces supporting a Φ-Poincaré inequality, as an extension of Farnana (Nonlinear Anal. 73 (2010), pp. 2819–2830).

1. Introduction

Shanmugalingam [34] studied the p-Dirichlet energy integral in metric measure spaces $X = (X, d, \mu)$, and showed the existence of a minimizer in Newtonian space $N^{1,p}(X)$ which is defined in terms of p-weak upper gradients of functions in X. For basic properties of $N^{1,p}(X)$, see [33]. We refer to e.g. [10, 11, 16, 17, 24, 25, 31, 35] for Sobolev spaces on metric measure spaces. See Kinnunen-Martio [20] and Mocanu [27] for the single obstacle problem on Newtonian spaces.

Farnana [6] studied the double obstacle problem for p-Dirichlet energy integrals in $N^{1,p}(X)$. The double obstacle problem in $\mathbf{R}^N$ was studied in [4] for the case $p = 2$ and in [19, 22] for the case $p > 1$. For convergence properties of the obstacle problem in $\mathbf{R}^N$, see e.g. [21, 32]. Farnana [7] studied continuous dependence on obstacles for the double obstacle problem on metric measure spaces as an extension of [32], and studied generalized solutions of the double obstacle problem.

Variable exponent Lebesgue spaces, Musielak-Orlicz spaces and Sobolev spaces have attracted lots of attention to discuss nonlinear partial differential equations with non-standard growth conditions. For survey books, see [3, 5, 12]. Acerbi and Mingione [1] studied the existence and the regularity of min-

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imizers of the $p(\cdot)$-Dirichlet energy integral on a bounded domain in $\mathbf{R}^N$. Variable exponent Sobolev spaces with zero boundary values on $\mathbf{R}^N$ was studied in [13]. In the past two decades, variable exponent Sobolev spaces on metric measure spaces have been studied by many researchers, see e.g. [8, 14, 15, 26]. Let Ω be a measurable set in X. Musielak-Orlicz Newtonian spaces $N^{1,\Phi}(\Omega)$ on X defined by a function $\Phi(x,t) : X \times [0,\infty) \to [0,\infty)$ were introduced in [29]. In [30], Musielak-Orlicz-Sobolev spaces with zero boundary values on X were studied, as an extension of [13, 18]. In [23], the single obstacle problems for Musielak-Orlicz Dirichlet energy integral on X were discussed.

In the previous paper [9], we proved the existence and uniqueness of a solution to the double obstacle problem for a Φ-Dirichlet energy integral on a bounded open set in X, as an extension of [6, 13, 23]. In [9], we also showed the solution u of the double obstacle problem with obstacles ψ and φ can be obtained as the limit of the solutions u_j of the double obstacle problem with obstacles ψ_j and φ_j converging to ψ and φ respectively.

In the present paper, based on the idea by Farnana [6], we introduce generalized solutions of the $\{\psi,\varphi\}$-problem in Ω for boundary values $f \notin N^{1,\Phi}(\Omega)$ or in the case where there is no Newtonian function between the obstacles ψ and φ with the given boundary values f. We prove the existence and uniqueness of a generalized solution to the double obstacle problem for a Φ-Dirichlet energy integral on a bounded open set in X (Theorem 3.4), as an extension of [7, Theorem 4.4].

We also prove that generalized solutions u of the $\{\psi,\varphi\}$-problem in Ω is locally a solution of the $\mathscr{K}_{\psi,\varphi,u}$-obstacle problem in $N^{1,\Phi}$ and that $u \in N^{1,\Phi}_{\mathrm{loc}}(\Omega)$ provided the two obstacles ψ and φ are separated by a Newtonian function (Theorem 3.7), as an extension of [7, Theorem 4.10].

Throughout this paper, let C denote various constants independent of the variables in question and $C(a,b,\cdots)$ be a constant that depends on $a,b,\ldots$.

2. Notation and preliminaries

We denote by (X,d,μ) a metric measure space, where X is a set, d is a metric on X and μ is a nonnegative complete Borel regular outer measure on X which is finite and positive for every open ball in X. For simplicity, we often write X instead of (X,d,μ). For $x \in X$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x with radius r. We denote by χ_E the characteristic function of $E \subset X$.

We consider a function

$$\Phi(x,t) : X \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions ($\Phi1$)–($\Phi4$):

($\Phi1$) $\Phi(\cdot, t)$ is measurable on X for each $t \geq 0$ and $\Phi(x, \cdot)$ is continuous on $[0, \infty)$ for each $x \in X$;

($\Phi2$) $\Phi(x, 0) = 0$ and $\Phi(x, \cdot)$ is a convex function on $[0, \infty)$ for every $x \in X$;

($\Phi3$) $0 < \inf_{x \in B} \Phi(x, 1) \leq \sup_{x \in B} \Phi(x, 1) < \infty$ for every open ball B in X;

($\Phi4$) there exists a constant $A_d \geq 2$ such that

$$\Phi(x, 2t) \leq A_d \Phi(x, t) \qquad \text{for all } x \in X \text{ and } t > 0.$$

Note from ($\Phi2$) that $\Phi(x, \cdot)$ is increasing on $[0, \infty)$ for every $x \in X$. Further, note that ($\Phi2$) and ($\Phi4$) imply

$$a\Phi(x, t) \leq \Phi(x, at) \leq \frac{A_d}{2} a^{\log_2 A_d} \Phi(x, t) \qquad \text{for } a \geq 1. \tag{2.1}$$

For an example of $\Phi(x, t)$ satisfying ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$), see [23, Example 2.3].

Let Ω be a measurable set in X. For $\Phi(x, t)$ satisfying ($\Phi1$), ($\Phi2$), ($\Phi3$) and ($\Phi4$), the associated Musielak-Orlicz space

$$L^{\Phi}(\Omega) = \left\{ f : f \text{ is a measurable function on } \Omega \text{ such that } \int_{\Omega} \Phi(y, |f(y)|) d\mu(y) < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{ \lambda > 0; \int_{\Omega} \Phi(y, |f(y)|/\lambda) d\mu(y) \leq 1 \right\}$$

if we identify functions which are equal μ-a.e. (cf. [28]).

For a function $u : \Omega \to [-\infty, \infty]$, a nonnegative measurable function h on Ω is said to be a Φ-weak upper gradient of u in Ω if

$$|u(\gamma(0)) - u(\gamma(\ell_\gamma))| \leq \int_{\gamma} h \, ds \tag{2.2}$$

holds for M_Φ-a.e. $\gamma \in \Gamma(\Omega)$, where $\Gamma(\Omega)$ is the family of all rectifiable curves $\gamma : [0, \ell_\gamma] \to \Omega$ parameterized by arc length ds. Here, by saying that (2.2) holds, we understand that $\int_\gamma h \, ds$ is well-defined and $\int_\gamma h \, ds = \infty$ in case $|u(\gamma(0))| = \infty$ or $|u(\gamma(\ell_\gamma))| = \infty$ (cf. [2]). See [23] for the notion "M_Φ-a.e.".

The Musielak-Orlicz Newtonian space $N^{1,\Phi}(\Omega)$ is defined to be the family of all $u \in L^{\Phi}(\Omega)$ having a Φ-weak upper gradient $h \in L^{\Phi}(\Omega)$ in Ω. For $u \in N^{1,\Phi}(\Omega)$ we define

$$\|u\|_{N^{1,\Phi}(\Omega)} = \|u\|_{L^{\Phi}(\Omega)} + \inf_h \|h\|_{L^{\Phi}(\Omega)},$$

where the infimum is taken over all Φ-weak upper gradients h of u in Ω.

We say that $h_u \in L^{\Phi}(\Omega)$ is a minimal Φ-weak upper gradient of $u \in N^{1,\Phi}(\Omega)$ in Ω if h_u is a Φ-weak upper gradient of u in Ω and $h_u \le h$ μ-a.e. in Ω for all Φ-weak upper gradients $h \in L^{\Phi}(\Omega)$ of u in Ω. Note from [23, Lemma 3.6] that for $u \in N^{1,\Phi}(\Omega)$, there exists a minimal Φ-weak upper gradient h_u of u in Ω and h_u is unique up to sets of measure zero.

For $u \in N^{1,\Phi}(\Omega)$, we set

$$\hat{\rho}_{\Phi,\Omega}(u) = \int_{\Omega} \Phi(y, |u(y)|)\,d\mu(y) + \inf_h \int_{\Omega} \Phi(y, h(y))\,d\mu(y)$$

where the infimum is taken over all Φ-weak upper gradients h of u in Ω.

For $E \subset \Omega$, we denote

$$s_{\Phi}(E;\Omega) = \{u \in N^{1,\Phi}(\Omega) : u \ge 1 \text{ on } E\}$$

and define the Φ-capacity with respect to Ω by

$$c_{\Phi}(E;\Omega) = \inf_{u \in s_{\Phi}(E;\Omega)} \hat{\rho}_{\Phi,\Omega}(u).$$

In case $s_{\Phi}(E;\Omega) = \emptyset$, we set $c_{\Phi}(E;\Omega) = \infty$. If $X = \Omega$, we denote $s_{\Phi}(E;\Omega)$ and $c_{\Phi}(E;\Omega)$ by $s_{\Phi}(E)$ and $c_{\Phi}(E)$ respectively.

Note that $c_{\Phi}(\cdot;\Omega)$ is an outer measure; in particular, it is countably subadditive (see [29, Proposition 4.5]). For $E \subset \Omega$, $c_{\Phi}(E;\Omega) \le c_{\Phi}(E)$. See [23, Remark 4.2].

For a set $E \subset \Omega$, we say that a property holds $c_{\Phi}(\cdot;\Omega)$-q.e. in E, if it holds on E except of a set $F \subset E$ with $c_{\Phi}(F;\Omega) = 0$, where q.e. stands for quasi-everywhere.

If $u, v \in N^{1,\Phi}(\Omega)$ and $u = v$ μ-a.e. in Ω, then $u = v$ $c_{\Phi}(\cdot;\Omega)$-q.e. in Ω. Moreover, if Ω is an open set in X, then $u = v$ c_{Φ}-q.e. in Ω. See [23, Lemma 4.5].

We say that a function u is c_{Φ}-quasicontinuous on E if, for any $\varepsilon > 0$, there is an open set G such that $c_{\Phi}(G) < \varepsilon$ and $u|_{E \setminus G}$ is continuous.

REMARK 2.1. *If X is proper and continuous functions in X are dense in $N^{1,\Phi}(X)$, then every $u \in N^{1,\Phi}_{\mathrm{loc}}(\Omega)$ is c_{Φ}-quasicontinuous in an open set Ω and c_{Φ} is an outer capacity. The proof can be carried out along the lines in the proof of* [2, Theorems 5.29 and 5.31].

For $E \subset X$, we define

$$N_0^{1,\Phi}(E) = \{f|_E : f \in N^{1,\Phi}(X) \text{ and } f = 0 \text{ in } X \backslash E\}.$$

By [23, Lemma 4.4], we have

$$N_0^{1,\Phi}(E) = \{f|_E : f \in N^{1,\Phi}(X) \text{ and } f = 0 \ c_\Phi\text{-q.e. in } X \backslash E\}.$$

See also [23, Lemma 5.1].

We say that X supports a Φ-Poincaré inequality if, for every open ball B in X, there exist constants $C_P(B) > 0$ and $\lambda \geq 1$ such that

$$\|u - u_B\|_{L^\Phi(B)} \leq C_P(B)\|h\|_{L^\Phi(\lambda B)}$$

holds whenever h is a Φ-weak upper gradient of u on λB and u is integrable on B, where $u_B = \fint_B u \, d\mu$ is the mean-value of u on B. For an example, see [9, Example 2.6].

From now on, we assume that Ω is a bounded open set with $c_\Phi(X \backslash \Omega) > 0$.

For $f \in N^{1,\Phi}(\Omega)$ and $\psi, \varphi : \Omega \to [-\infty, \infty]$, we define

$$\mathscr{K}_{\psi,\varphi,f}(\Omega) = \{u \in N^{1,\Phi}(\Omega) : u - f \in N_0^{1,\Phi}(\Omega) \text{ and } \psi \leq u \leq \varphi \ c_\Phi\text{-q.e. in } \Omega\}.$$

A function $u \in \mathscr{K}_{\psi,\varphi,f}(\Omega)$ is called a solution of the $\mathscr{K}_{\psi,\varphi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ if

$$\int_\Omega \Phi(x, h_u(x)) d\mu(x) \leq \int_\Omega \Phi(x, h_v(x)) d\mu(x)$$

for all $v \in \mathscr{K}_{\psi,\varphi,f}(\Omega)$.

We shall need the following result from [9, Theorem 3.1], which is a generalization of [6, 23].

Theorem 2.2. *Assume that $L^\Phi(\Omega)$ is reflexive and X supports a Φ-Poincaré inequality. Let $f \in N^{1,\Phi}(\Omega)$ and $\psi, \varphi : \Omega \to [-\infty, \infty]$. If $\mathscr{K}_{\psi,\varphi,f}(\Omega) \neq \emptyset$, then there exists a solution of the $\mathscr{K}_{\psi,\varphi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$.*

Further, if $\Phi(x, \cdot)$ is strictly convex for μ-a.e. $x \in \Omega$, then the solution of the $\mathscr{K}_{\psi,\varphi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ is unique (up to sets of c_Φ-capacity zero).

From now on we assume that $L^\Phi(\Omega)$ is reflexive, X supports a Φ-Poincaré inequality and $\Phi(x, \cdot)$ is strictly convex for μ-a.e. $x \in \Omega$.

We need the following comparison principle from [9, Lemma 3.3].

Lemma 2.3. *Let $f, f' \in N^{1,\Phi}(\Omega)$ and $\psi, \psi', \varphi, \varphi' : \Omega \to [-\infty, \infty]$. Assume that $\psi \leq \psi'$ and $\varphi \leq \varphi'$ c_Φ-q.e. in Ω and that $(f - f')_+ \in N_0^{1,\Phi}(\Omega)$. Let u be a solution of the $\mathscr{K}_{\psi,\varphi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ and u' be a solution of the $\mathscr{K}_{\psi',\varphi',f'}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$. Then $u \leq u'$ c_Φ-q.e. in Ω.*

The following lemma is from [9, Lemma 5.1].

LEMMA 2.4. *Suppose* $\{u_j\}$ *is a bounded sequence in* $N^{1,\Phi}(\Omega)$ *and* $u_j \to u$ c_Φ*-q.e. in* Ω. *Then* $u \in N^{1,\Phi}(\Omega)$ *and*

$$\int_\Omega \Phi(x, h_u(x))d\mu(x) \le \liminf_{j\to\infty} \int_\Omega \Phi(x, h_{u_j}(x))d\mu(x). \tag{2.3}$$

3. Generalized solutions

In this section, we assume that X is proper and continuous functions in X are dense in $N^{1,\Phi}(X)$. We say that $w_j \to w$ c_Φ-q.e. uniformly in Ω if there exists a set $E \subset \Omega$ such that $c_\Phi(E) = 0$ and $w_j \to w$ uniformly in $\Omega \setminus E$.

We say that u is a generalized solution of the $\{\psi, \varphi\}$-problem in Ω if there exist three sequences of functions $\{\psi_j\}_{j=1}^\infty$, $\{\varphi_j\}_{j=1}^\infty$ and $\{u_j\}_{j=1}^\infty$ such that ψ, φ and u are the c_Φ-q.e. uniform limits in Ω of ψ_j, φ_j and u_j respectively, and for every $j \in \mathbf{N}$ the function u_j is a solution of the $\mathscr{K}_{\psi_j,\varphi_j,u_j}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$.

It is clear that if u is a generalized solution of the $\{\psi, \varphi\}$-problem in Ω, then u is c_Φ-quasicontinuous in Ω by Remark 2.1, $\psi \le u \le \varphi$ c_Φ-q.e. in Ω and u is a generalized solution of the $\{\psi, \varphi\}$-problem in Ω' for every $\Omega' \subset\subset \Omega$ by [9, Lemma 4.6].

The following lemma is needed.

LEMMA 3.1 (cf. [7, Lemma 4.2]). *Let* $f_j, f \in N^{1,\Phi}(\Omega)$ *and* $\psi_j, \varphi_j, \psi, \varphi : \Omega \to [-\infty, \infty]$, $j = 1, 2, \dots,$ *be such that* $f_j \to f$, $\psi_j \to \psi$ *and* $\varphi_j \to \varphi$ c_Φ*-q.e. uniformly in* Ω. *Let also* u_j *be a solution of the* $\mathscr{K}_{\psi_j,\varphi_j,f_j}(\Omega)$*-obstacle problem in* $N^{1,\Phi}(\Omega)$, $j = 1, 2, \dots,$ *and* u *be a solution of the* $\mathscr{K}_{\psi,\varphi,f}(\Omega)$*-obstacle problem in* $N^{1,\Phi}(\Omega)$. *Then* $u_j \to u$ c_Φ*-q.e. uniformly in* Ω.

PROOF. Let $\varepsilon > 0$. Then there exist a set $E \subset \Omega$ and a number $j_0 \in \mathbf{N}$ such that $c_\Phi(E) = 0$ and $\psi - \varepsilon \le \psi_j \le \psi + \varepsilon$, $\varphi - \varepsilon \le \varphi_j \le \varphi + \varepsilon$, $f - \varepsilon \le f_j \le f + \varepsilon$ on $\Omega \setminus E$ for every $j \ge j_0$. Since $u + \varepsilon$ is a solution of the $\mathscr{K}_{\psi+\varepsilon,\varphi+\varepsilon,f+\varepsilon}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ and $u - \varepsilon$ is a solution of the $\mathscr{K}_{\psi-\varepsilon,\varphi-\varepsilon,f-\varepsilon}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$, Lemma 2.3 shows that $u - \varepsilon \le u_j \le u + \varepsilon$ c_Φ-q.e. in Ω. Thus $u_j \to u$ c_Φ-q.e. uniformly in Ω. □

LEMMA 3.2 (cf. [2, Theorem 2.36]). *The space* $N_0^{1,\Phi}(\Omega)$ *is a closed subspace of* $N^{1,\Phi}(\Omega)$.

PROOF. Let $u_j \in N_0^{1,\Phi}(\Omega)$ for each $j \in \mathbf{N}$ and $u \in N^{1,\Phi}(\Omega)$ such that $u_j \to u$ in $N^{1,\Phi}(\Omega)$. Then $u_j \to v$ in $N^{1,\Phi}(X)$ for some $v \in N^{1,\Phi}(X)$ with $v = u$ c_Φ-q.e. in Ω as we can consider u_j to be identically zero outside Ω. Since there exists a subsequence of $\{u_j\}_{j=1}^{\infty}$ which converges to v pointwise c_{Φ}-q.e. in X, $v=0$ c_{Φ}-q.e. in $X\backslash\Omega$, so that, $u \in N_0^{1,\Phi}(\Omega)$. □

Lemma 3.3 (cf. [7, Lemma 4.3]). *Let $u \in N^{1,\Phi}(\Omega)$. Assume that there exists a c_{Φ}-quasicontinuous function $f : \bar{\Omega} \to [-\infty, \infty]$ such that $u \leq f$ c_{Φ}-q.e. in Ω and $f=0$ c_{Φ}-q.e. on $\partial\Omega$. Then $u_+ = \max\{u, 0\} \in N_0^{1,\Phi}(\Omega)$.*

Proof. By replacing u and f by u_+ and f_+ respectively if necessary we may assume that $u \geq 0$ and $f \geq 0$. Assume that $0 \leq u \leq f \leq 1$ c_{Φ}-q.e. in Ω. Since f is c_{Φ}-quasicontinuous in $\bar{\Omega}$, for every $j \in \mathbf{N}$ there exists an open set G_j such that $f|_{\bar{\Omega}\backslash G_j}$ is continuous and $c_{\Phi}(G_j) < 1/2^j$. By the definition of capacity we can find a decreasing sequence of nonnegative functions $\{\eta_j\}_{j=1}^{\infty}$ such that $\hat{\rho}_{\Phi,X}(\eta_j) < 1/2^{j-2}$ and $\eta_j \geq 1$ in G_j. Since $\eta_j \to 0$ in $N^{1,\Phi}(X)$, replacing $\{\eta_j\}_{j=1}^{\infty}$ by a subsequence if necessary, we may assume that $\eta_j \to 0$ c_{Φ}-q.e. in X. Let

$$u_j = \max\{u - 1/j - \eta_j, 0\}.$$

Then $u_j \in N^{1,\Phi}(\Omega)$ for each $j \in \mathbf{N}$. Note that, as $f=0$ c_{Φ}-q.e. on $\partial\Omega$, we may assume that $f(x) = 0$ for every $x \in \partial\Omega\backslash G_j$. Then, for every $j \in \mathbf{N}$, the set

$$F_j = \{x \in \bar{\Omega} : f(x) \geq 1/j\}\backslash G_j$$

is compact and contained in Ω.

Next we show that $u_j \in N_0^{1,\Phi}(\Omega)$. To this end note first that

$$\Omega\backslash F_j = \{x \in \Omega : f(x) < 1/j\} \cup (G_j \cap \Omega).$$

Then for c_{Φ}-q.e. $x \in \{x \in \Omega : f(x) < 1/j\}$ we have $u(x) \leq f(x) < 1/j$. Thus

$$u(x) - 1/j - \eta_j(x) < -\eta_j(x) \leq 0$$

and hence $u_j(x) = 0$. If c_{Φ}-q.e. $x \in G_j \cap \Omega$ then we get that

$$u(x) \leq 1 \leq \eta_j(x) \leq \eta_j(x) + 1/j$$

which implies that $u_j(x) = 0$. Then we conclude that $u_j = 0$ c_{Φ}-q.e. on $\Omega\backslash F_j$ and hence $u_j \in N_0^{1,\Phi}(\Omega)$. We will show below that $u_j \to u$ in $N^{1,\Phi}(\Omega)$ which shows that $u \in N_0^{1,\Phi}(\Omega)$ by Lemma 3.2.

To show that $u_j \to u$ in $N^{1,\Phi}(\Omega)$, let

$$A_j = \{x \in \Omega : 0 < u(x) < \eta_j(x) + 1/j\}$$

and

$$B_j = \{x \in \Omega : u(x) \geq \eta_j(x) + 1/j\}.$$

Then we have

$$u_j - u = \begin{cases} -u & \text{in } A_j, \\ 0 & \text{in } \{x \in \Omega : u(x) = 0\}, \\ -1/j - \eta_j & \text{in } B_j. \end{cases}$$

Since there is a set $E \subset \Omega$ such that $c_\Phi(E) = 0$ and $\eta_j \to 0$ in $\Omega \setminus E$ we get that $\bigcap_{j=1}^{\infty} A_j \setminus E = \emptyset$ and $\mu(A_j) \to 0$ as $j \to \infty$. The dominated convergence theorem and the fact that $\eta_j \to 0$ in $N^{1,\Phi}(\Omega)$ imply that

$$\begin{aligned} &\int_\Omega \Phi(x, u_j(x) - u(x)) d\mu(x) \\ &\quad = \int_{A_j} \Phi(x, u(x)) d\mu(x) + \int_{B_j} \Phi(x, \eta_j(x) + 1/j) d\mu(x) \\ &\quad \le \int_{A_j} \Phi(x, u(x)) d\mu(x) + A_d \left(\int_\Omega \Phi(x, \eta_j(x)) d\mu(x) + \frac{1}{j} \int_\Omega \Phi(x, 1) d\mu(x) \right) \\ &\quad \to 0 \end{aligned}$$

as $j \to \infty$ by ($\Phi 4$) and ($\Phi 3$) and

$$\begin{aligned} &\int_\Omega \Phi(x, h_{u_j - u}(x)) d\mu(x) \\ &\qquad = \int_{A_j} \Phi(x, h_u(x)) d\mu(x) + \int_{B_j} \Phi(x, h_{\eta_j}(x)) d\mu(x) \to 0 \end{aligned}$$

as $j \to \infty$. Thus $u_j \to u$ in $N^{1,\Phi}(\Omega)$ and hence $u \in N_0^{1,\Phi}(\Omega)$.

Finally if f is unbounded, then for every $k \in \mathbf{N}$ we have $0 \le \min\{u, k\} \le \min\{f, k\}$ and the above argument shows that $\min\{u, k\} \in N_0^{1,\Phi}(\Omega)$ for all $k \in \mathbf{N}$. As $\min\{u, k\} \to u$ in $N^{1,\Phi}(\Omega)$ we get that $u \in N_0^{1,\Phi}(\Omega)$. □

We shall show an existence and uniqueness result for generalized solutions of the double obstacle problem, which is a generalization of [7, Theorem 4.4].

THEOREM 3.4. *Let $\psi, \varphi : \Omega \to [-\infty, \infty]$ be such that $\psi \le \varphi$ c_Φ-q.e. in Ω and $f : \overline{\Omega} \to [-\infty, \infty]$ be a c_Φ-quasicontinuous function on $\overline{\Omega}$ such that $\psi \le f \le \varphi$ c_Φ-q.e. in Ω. Assume that there exist $f_j \in N^{1,\Phi}(\overline{\Omega})$ such that f_j is a c_Φ-quasicontinuous function on $\overline{\Omega}$ and $f_j \to f$ c_Φ-q.e. uniformly in $\overline{\Omega}$. Then there exists a unique up to sets of c_Φ-capacity zero, c_Φ-quasicontinuous function $u : \overline{\Omega} \to [-\infty, \infty]$ that is a generalized solution of the $\{\psi, \varphi\}$-problem in Ω and is such that $u = f$ c_Φ-q.e. on $\partial\Omega$.*

REMARK 3.5. *Let $f \in N^{1,\Phi}(\overline{\Omega})$ be a c_Φ-quasicontinuous function on $\overline{\Omega}$ and let u be a solution of the $\mathscr{K}_{\psi,\varphi,f}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$. Let $u = f$*

on $\partial\Omega$. *Then* $u \in N^{1,\Phi}(\bar{\Omega})$ *and* u *is a* c_Φ*-quasicontinuous function on* $\bar{\Omega}$ *by Remark* 2.1.

Proof of Theorem 3.4. Since $f_j \to f$ c_Φ-q.e. uniformly in $\bar{\Omega}$, there exists an increasing sequence $\{k_j\}_{j=1}^{\infty}$ such that $|f_{k_j} - f| < 2^{-3-j}$ c_Φ-q.e. in $\bar{\Omega}$. Let $\tilde{f}_j = f_{k_j} + 2^{-1-j}$. Then we see that $\tilde{f}_j \in N^{1,\Phi}(\bar{\Omega})$, $\tilde{f}_j$ decreases c_Φ-q.e. uniformly to f in $\bar{\Omega}$ and $0 \le \tilde{f}_j - f \le 2^{-j}$ c_Φ-q.e. in $\bar{\Omega}$. Hence we may assume without loss of generality that f_j decreases c_Φ-q.e. uniformly to f in $\bar{\Omega}$ and $0 \le f_j - f \le 2^{-j}$ c_Φ-q.e. in $\bar{\Omega}$. It follows that

$$\psi \le f \le f_j \le f + 2^{-j} \le \varphi + 2^{-j} \qquad c_\Phi\text{-q.e. in } \Omega.$$

Since $f_j \in \mathscr{K}_{\psi,\varphi+2^{-j},f_j}(\Omega)$, there exists a solution u_j of the $\mathscr{K}_{\psi,\varphi+2^{-j},f_j}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ by Theorem 2.2. Let $u_j = f_j$ on $\partial\Omega$. Then u_j is c_Φ-quasicontinuous on $\bar{\Omega}$ by Remark 3.5. Fix $k \in \mathbf{N}$. Since $\varphi + 2^{-j} \le \varphi + 2^{-k}$ and $f_j \le f_k$ c_Φ-q.e. in Ω for all $j \ge k$, Lemma 2.3 implies that for all $j \ge k$

$$u_j \le u_k \qquad c_\Phi\text{-q.e. in } \Omega. \tag{3.1}$$

Further, we see that $u_j + 2^{-k}$ is a solution of the $\mathscr{K}_{\psi+2^{-k},\varphi+2^{-j}+2^{-k},f_j+2^{-k}}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$ and $f_k \le f + 2^{-k} \le f_j + 2^{-k}$ c_Φ-q.e. in Ω. Lemma 2.3 again implies that for all $j \ge k$

$$u_k \le u_j + 2^{-k} \qquad c_\Phi\text{-q.e. in } \Omega. \tag{3.2}$$

Together with $u_j = f_j \le f_k = u_k \le f + 2^{-k} \le f_j + 2^{-k} = u_j + 2^{-k}$ c_Φ-q.e. in $\partial\Omega$ for all $j \ge k$, (3.1) and (3.2) imply that for all $j \ge k$

$$u_j \le u_k \le u_j + 2^{-k} \qquad c_\Phi\text{-q.e. in } \bar{\Omega}. \tag{3.3}$$

It follows from (3.3) that $u_1 \ge u_2 \ge \cdots$ c_Φ-q.e. in $\bar{\Omega}$. Let $u(x) = \lim_{j\to\infty} u_j(x)$ for c_Φ-q.e. $x \in \bar{\Omega}$ and define u arbitrarily elsewhere. Then letting $j \to \infty$ in (3.3), we get that $u \le u_k \le u + 2^{-k}$ c_Φ-q.e. in $\bar{\Omega}$. This shows that $u_k \to u$ c_Φ-q.e. uniformly in $\bar{\Omega}$ and u is c_Φ-quasicontinuous on $\bar{\Omega}$.

We next prove the uniqueness. Assume that u_1 and u_2 are generalized solutions of the $\{\psi,\varphi\}$-problem in Ω such that u_1, u_2 are c_Φ-quasicontinuous on $\bar{\Omega}$ and $u_1 = u_2 = f$ c_Φ-q.e. on $\partial\Omega$. By definition there exist six sequences $\{\psi_{1,j}\}_{j=1}^{\infty}$, $\{\varphi_{1,j}\}_{j=1}^{\infty}$, $\{u_{1,j}\}_{j=1}^{\infty}$, $\{\psi_{2,j}\}_{j=1}^{\infty}$, $\{\varphi_{2,j}\}_{j=1}^{\infty}$ and $\{u_{2,j}\}_{j=1}^{\infty}$ such that $u_{1,j}$ is a solution of the $\mathscr{K}_{\psi_{1,j},\varphi_{1,j},u_{1,j}}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$, $u_{2,j}$ is a solution of the $\mathscr{K}_{\psi_{2,j},\varphi_{2,j},u_{2,j}}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$, and $\psi_{1,j} \to \psi$, $\varphi_{1,j} \to \varphi$, $u_{1,j} \to u_1$, $\psi_{2,j} \to \psi$, $\varphi_{2,j} \to \varphi$ and $u_{2,j} \to u_2$ c_Φ-q.e. uniformly in Ω. We may assume without loss of generality that $|\psi_{1,j} - \psi_{2,j}| \le 2^{-j}$, $|\varphi_{1,j} - \varphi_{2,j}| \le 2^{-j}$,

$|u_{1,j} - u_1| \le 2^{-j}$ and $|u_{2,j} - u_2| \le 2^{-j}$ c_Φ-q.e. in Ω. It follows that

$$u_{2,j} - u_{1,j} - 2^{1-j} \le |u_{2,j} - u_2| + |u_2 - u_1| + |u_1 - u_{1,j}| - 2^{1-j} \le |u_2 - u_1|$$

c_Φ-q.e. in Ω. As $|u_2 - u_1|$ is c_Φ-quasicontinuous on $\bar{\Omega}$ and $|u_2 - u_1| = 0$ c_Φ-q.e. on $\partial\Omega$, Lemma 3.3 shows that $(u_{2,j} - u_{1,j} - 2^{1-j})_+ \in N_0^{1,\Phi}(\Omega)$. Further, we see that $u_{1,j} + 2^{1-j}$ is a solution of the $\mathscr{K}_{\psi_{1,j}+2^{1-j},\varphi_{1,j}+2^{1-j},u_{1,j}+2^{1-j}}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$, $\psi_{2,j} \le \psi_{1,j} + 2^{1-j}$ and $\varphi_{2,j} \le \varphi_{1,j} + 2^{1-j}$ c_Φ-q.e. in Ω. Hence we obtain by Lemma 2.3

$$u_{2,j} \le u_{1,j} + 2^{1-j}$$

c_Φ-q.e. in Ω. Letting $j \to \infty$ we get $u_2 \le u_1$ c_Φ-q.e. in Ω. Similarly we get $u_1 \le u_2$ c_Φ-q.e. in Ω, and hence $u_1 = u_2$ c_Φ-q.e. in Ω. □

LEMMA 3.6 (cf. [7, Remark 4.7]). *Let u be a generalized solution of the $\{\psi,\varphi\}$-problem in Ω. For every open set $\Omega' \subset\subset \Omega$, there exists a sequence $\{u_j\}_{j=1}^\infty$ such that $u_j \in N^{1,\Phi}(\Omega')$ is a solution of the $\mathscr{K}_{\psi,\varphi+2^{-j},u_j}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$ and u_j decreases to u c_Φ-q.e. uniformly in Ω'.*

PROOF. By definition there exist three sequences of functions $\{\psi_j\}_{j=1}^\infty$, $\{\varphi_j\}_{j=1}^\infty$ and $\{\tilde{u}_j\}_{j=1}^\infty$ such that ψ, φ and u are the c_Φ-q.e. uniform limits in Ω of ψ_j, φ_j and $\tilde{u}_j$ respectively, and for every $j \in \mathbf{N}$ the function $\tilde{u}_j$ is a solution of the $\mathscr{K}_{\psi_j,\varphi_j,\tilde{u}_j}(\Omega)$-obstacle problem in $N^{1,\Phi}(\Omega)$. By [9, Lemma 4.6], $\tilde{u}_j \in N^{1,\Phi}(\overline{\Omega'})$ is a solution of the $\mathscr{K}_{\psi_j,\varphi_j,\tilde{u}_j}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$ for every open set $\Omega' \subset\subset \Omega$. Then the proof of Theorem 3.4 with $\Omega = \Omega'$, $f_j = \tilde{u}_j$ and $f = u$ implies that there exist a solution u_j of the $\mathscr{K}_{\psi,\varphi+2^{-j},u_j}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$, $j = 1, 2, \dots$, and a generalized solution v of the $\{\psi,\varphi\}$-problem in Ω' such that u_j decreases to v c_Φ-q.e. uniformly in Ω' and $v = u$ c_Φ-q.e. on $\partial\Omega'$. Since u is a generalized solution of the $\{\psi,\varphi\}$-problem in Ω', we have $v = u$ c_Φ-q.e. in Ω' by uniqueness of Theorem 3.4. □

We shall show that if the two obstacles are separated by a Newtonian function then, locally, the generalized solution is the solution by Theorem 2.2.

THEOREM 3.7. *Let $\psi, \varphi : \Omega \to [-\infty, \infty]$ be two functions such that there exists $v \in N_{\mathrm{loc}}^{1,\Phi}(\Omega)$ with $\psi \le v \le \varphi$ c_Φ-q.e. in Ω. Let u be a generalized solution of the $\{\psi,\varphi\}$-problem in Ω. Then $u \in N_{\mathrm{loc}}^{1,\Phi}(\Omega)$ and u is a solution of the $\mathscr{K}_{\psi,\varphi,u}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$ for all $\Omega' \subset\subset \Omega$.*

PROOF. For $\Omega' \subset\subset \Omega$, Lemma 3.6 implies that there exists a sequence $\{u_j\}_{j=1}^\infty$ such that $u_j \in N^{1,\Phi}(\Omega')$ is a solution of the $\mathscr{K}_{\psi,\varphi+2^{-j},u_j}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$ and u_j decreases to u c_Φ-q.e. uniformly in Ω'. As Ω' is bounded we have $u_j \to u$ in $L^\Phi(\Omega')$ and hence $\{u_j\}_{j=1}^\infty$ is bounded in $L^\Phi(\Omega')$.

If we can show that $\{h_{u_j}\}_{j=1}^{\infty}$ is bounded in $L^{\Phi}(B)$ for all balls $B \subset\subset \Omega$ then Lemma 2.4 implies that $u \in N^{1,\Phi}_{\text{loc}}(\Omega)$.

To this end, let $B = B(x_0, R) \subset\subset B' = B(x_0, R') \subset \Omega'$ such that $R' \leq 1$. Let next $0 < r_1 < r_2 \leq R'$, $B_j = B(x_0, r_j)$, $j = 1, 2$, and

$$\eta(x) = \min\left\{\frac{r_2 - d(x_0, x)}{r_2 - r_1}, 1\right\}_+ \in N_0^{1,\Phi}(B_2).$$

Note that $\chi_{B_1} \leq \eta \leq 1$ and

$$h_\eta \leq \frac{1}{r_2 - r_1}\chi_{B_2 \setminus B_1}.$$

Set $v_j = \eta v + (1 - \eta)u_j = u_j + \eta(v - u_j) \in N^{1,\Phi}(B')$. By [2, Lemma 2.18], we have that

$$h_{v_j} \leq (1 - \eta)h_{u_j} + \eta h_v + |v - u_j| h_\eta$$

μ-a.e. in B'. Further, since $\psi \leq v \leq \varphi$ and $\psi \leq u_j \leq \varphi + 2^{-j}$, we have $\psi \leq v_j \leq \varphi + 2^{-j}$. This together with the fact that $v_j = u_j$ on ∂B_2 implies that $v_j \in \mathscr{K}_{\psi, \varphi+2^{-j}, u_j}(B_2)$. Using the fact that u_j is a solution of the $\mathscr{K}_{\psi, \varphi+2^{-j}, u_j}(B_2)$-obstacle problem in $N^{1,\Phi}(B_2)$ and $(\Phi 4)$, we have that

$$\begin{aligned}
&\int_{B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \\
&\quad \leq \int_{B_2} \Phi(x, h_{u_j}(x))\,d\mu(x) \\
&\quad \leq \int_{B_2} \Phi(x, h_{v_j}(x))\,d\mu(x) \\
&\quad \leq A_d^2\left(\int_{B_2} \Phi(x, (1 - \eta(x))h_{u_j}(x))\,d\mu(x) + \int_{B_2} \Phi(x, |v(x) - u_j(x)| h_\eta(x))\,d\mu(x)\right. \\
&\qquad \left. + \int_{B_2} \Phi(x, \eta(x) h_v(x))\,d\mu(x)\right) \\
&\quad \leq A_d^2\left(\int_{B_2 \setminus B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) + \int_{B_2} \Phi(x, |v(x) - u_j(x)|/(r_2 - r_1))\,d\mu(x)\right. \\
&\qquad \left. + \int_{B_2} \Phi(x, h_v(x))\,d\mu(x)\right).
\end{aligned}$$

Hence, by (2.1) and the fact that $\{u_j\}_{j=1}^{\infty}$ is bounded in $L^{\Phi}(\Omega')$

$$\begin{aligned}
&\int_{B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \\
&\quad \le A_d^2 \Bigg(\int_{B_2 \setminus B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \\
&\qquad + \frac{A_d}{2(r_2 - r_1)^{\log_2 A_d}} \int_{\Omega'} \Phi(x, |v(x) - u_j(x)|)\,d\mu(x) \\
&\qquad + \int_{\Omega'} \Phi(x, h_v(x))\,d\mu(x) \Bigg) \\
&\quad \le A_d^2 \left(\int_{B_2 \setminus B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) + \frac{C_1}{(r_2 - r_1)^{\log_2 A_d}} + C_2 \right).
\end{aligned}$$

Adding A_d^2 times the left-hand side to both sides we obtain

$$\begin{aligned}
&(1 + A_d^2) \int_{B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \\
&\quad \le A_d^2 \left(\int_{B_2} \Phi(x, h_{u_j}(x))\,d\mu(x) + \frac{C_1}{(r_2 - r_1)^{\log_2 A_d}} + C_2 \right).
\end{aligned}$$

After dividing by $1 + A_d^2$ we get, with $\theta = A_d^2/(1 + A_d^2) < 1$, that

$$\int_{B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \le \theta \int_{B_2} \Phi(x, h_{u_j}(x))\,d\mu(x) + \frac{C_1\theta}{(r_2 - r_1)^{\log_2 A_d}} + C_2\theta.$$

Applying [2, Lemma 7.18] we obtain that

$$\int_{B_1} \Phi(x, h_{u_j}(x))\,d\mu(x) \le C \left(\frac{C_1\theta}{(r_2 - r_1)^{\log_2 A_d}} + C_2\theta \right)$$

for $0 < r_1 < r_2 \le R'$. By choosing $r_1 = R$ and $r_2 = R'$ we see that $\{h_{u_j}\}_{j=1}^{\infty}$ is bounded in $L^{\Phi}(B)$. By Lemma 2.4, $u \in N^{1,\Phi}(B)$, and hence $u \in N^{1,\Phi}_{\mathrm{loc}}(\Omega)$.

Since $u \in \mathscr{K}_{\psi,\varphi,u}(\Omega')$, there exists a solution $\tilde{u}$ of the $\mathscr{K}_{\psi,\varphi,u}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$ by Theorem 2.2. Further, by Lemma 3.1, we have $u_j \to \tilde{u}$ c_{Φ}-q.e. uniformly in Ω', and hence $\tilde{u} = u$ c_{Φ}-q.e. in Ω' and u is a solution of the $\mathscr{K}_{\psi,\varphi,u}(\Omega')$-obstacle problem in $N^{1,\Phi}(\Omega')$. □

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