# Unified Description of Elastoplastic Deformation of Solids†

**–from Subloading Surface Concept to Tangential Plasticity–**

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**Abstract**

*The subloading surface model is based on the natural postulate that the plastic strain rate develops as the stress approaches the yield surface. Then, it is capable of describing rigorously the rate-independent/dependent elastoplastic deformation in monotonic/cyclic and proportional/non-proportional loadings. In addition, it is capable of describing rigorously the friction phenomenon between solids. Further, it provides a high efficiency in numerical calculation since the stress is attracted automatically to the yield surface. Then, the subloading surface model possesses the high capability of describing uniformly and rigorously the elastoplastic deformation and interaction of solids. This fact, mainly focused from the introduction of the subloading surface concept to the tangential plasticity, is analyzed and deliberated in this article.*

**KEY WORDS: (elastoplasticity), (constitutive equation), (cyclic plasticity), (friction theory), (rate-dependence), (subloading surface)**

## 1. Introduction

The deformation analysis of solids and structures with high accuracy and numerical efficiency is required increasingly to enhance their mechanical performance, strength and durability responding to the rapid development of engineering technologies. Various elastoplastic constitutive models have been proposed to this end. Here, it would be of the crucial importance to perceive the rigorous one among them and adopt it in the deformation analyses, which will be circulated widely with the passage of time and remain universally in the history of elastoplasticity for the sound development of elastoplasticity.

The mechanical requirements for the constitutive equation in rate form describing inelastic deformation, i.e. the continuity and the smoothness conditions (Hashiguchi, 1993a,b, 2000, 2013) are defined. In addition, the general loading criterion is deduced on the physical basis, which holds not only in the hardening state but also in the perfectly-plastic and the softening states. Then, the subloading surface model (Hashiguchi, 1980, 1989, 2013) in the framework of the hypoelastic-based plastic constitutive equation is shown, which is based on the natural postulate that the plastic strain rate develops as the stress approaches the yield surface, describing pertinently the plastic strain rate induced by the rate of stress inside the yield surface. Then, it is capable of describing appropriately the monotonic, the non-proportional and the cyclic loadings, the rate-dependent deformation behavior in a general rate up to the impact load for wide classes of materials, e.g. metals and soils and the friction phenomena between solids. Here, it satisfies the fundamental requirements for elastoplastic constitutive equations, i.e. the continuity and the smoothness conditions and possesses the stress controlling function to attract the stress to the yield surface. Eventually, the pertinence and the generality for descriptions of elastoplastic deformation behavior of solids and the adaptability to the numerical calculation are materialized in the subloading surface model. Therefore, the subloading surface model would be regarded to provide the basic structure of the elastoplastic deformation and interaction which has been studied over the last one century as will be described briefly in this article.

## 2. Basic structure of elastoplastic constitutive equation

The strain rate $\mathbf{d}$, i.e. the symmetric part of a velocity gradient is decomposed additively into the elastic strain rate $\mathbf{d}^e$ and the plastic strain rate $\mathbf{d}^p$, i.e.

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p \tag{1}$$

Firstly, the elastic strain rate is linearly related to the stress rate in the hypoelasticity as follows:

$$\mathbf{d}^e = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} \tag{2}$$

Where $\boldsymbol{\sigma}$ is the Cauchy stress, ( $\circ$ ) designating the corotational rate, and $\mathbf{E}$ is the elastic tangent modulus tensor and is given explicitly as

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$$\left.\begin{aligned}\mathbf{E} &= K\mathcal{T} + 2G\mathcal{I}' = \frac{nE}{(1+n)(1-2n)}\mathcal{I} + \frac{E}{1+n}\mathcal{S} \\ \mathbf{E}^{-1} &= \frac{1}{3K}\mathcal{T} + \frac{1}{2g}\mathcal{I}' = \frac{2-n}{3E}\mathcal{I} + \frac{1+n}{2E}\mathcal{S}\end{aligned}\right\} \quad (3)$$

where *K*, *G*, *E* and $\nu$ are the bulk, the shear, the Young's modulus and Poisson's ratio, respectively, which are functions of stress in general, and the fourth-order tensors defined in the following is exploited.

$$\mathcal{I} \equiv \delta_{ik}\delta_{jl}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_i \otimes \mathbf{e}_j \quad (4)$$

$$\underline{\mathcal{I}} \equiv \delta_{il}\delta_{jk}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \otimes \mathbf{e}_i \quad (5)$$

$$\mathcal{T} \equiv \delta_{ij}\delta_{kl}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathbf{e}_i \otimes \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_j \quad (6)$$

$$\mathcal{I}' \equiv (\delta_{ik}\delta_{jl} - \tfrac{1}{3}\delta_{ij}\delta_{kl})\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = \mathcal{I} - \tfrac{1}{3}\mathcal{T} \quad (7)$$

$$\mathcal{S} \equiv \tfrac{1}{2}(\mathcal{I} + \underline{\mathcal{I}}) \quad (8)$$

where $\delta_{ij}$ is the Kronecker's delta, i.e. $\delta_{ij} = 1$ for *i*=*j* and $\delta_{ij} = 0$ for $i \neq j$. The symbols $\mathcal{I}$, $\underline{\mathcal{I}}$, $\mathcal{T}$, $\mathcal{I}'$ and $\mathcal{S}$ stand for the fourth-order *identity*, *transposing*, *tracing-identity*, *deviatoric projection* and *symmeytrizing tensor*, respectively, which play the roles of the transformations $\mathcal{I}\mathbf{:t} = \mathbf{t}$, $\underline{\mathcal{I}}\mathbf{:t} = \mathbf{t}^T$, $\mathcal{I}'\mathbf{:t} = \mathbf{t}'$, $\mathcal{T}\mathbf{:t} = (\mathrm{tr}\,\mathbf{t})\mathbf{I}$ where $\mathcal{I}\mathbf{:t} = \mathbf{t}$, is the second-order identity tensor.

Then, the plastic strain rate is required to be formulated adequately.

The yield surface is given by

$$f(\hat{\boldsymbol{\sigma}}) = F(H) \quad (9)$$

where

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha} \quad (10)$$

$\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress.

## 3. Continuity and smoothness conditions

Before formulation of elastoplastic constitutive equations, the continuity and the smoothness conditions (Hashiguchi, 1993a,b, 2000, 2013) are described in this section. These conditions are required to formulate elastoplastic constitutive equations pertinently. In particular, they are of crucial importance for the description of cyclic loading behavior in which the delicate description of fine plastic strain rate induced by the rate of stress inside the yield surface is required.

### 3.1 Continuity condition

It is observed in experiments that "stress rate changes continuously for a continuous change of strain rate". This property is called the continuity condition and is expressed mathematically as follows.

$$\lim_{\delta\mathbf{d}\to\mathbf{O}} \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, H_i; \mathbf{d} + \delta\mathbf{d}) = \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, H_i; \mathbf{d}) \quad (11)$$

where $H_i$ (*i*=1, 2, 3,•••) denotes collectively scalar-valued or tensor-valued internal state variables and $\delta(\ )$stands for an infinitesimal variation. The response of the stress rate to the input of strain rate in the current state of stress and internal variables is designated by $\overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, H_i; \mathbf{d})$. Uniqueness of solution is not guaranteed in constitutive equations violating the continuity condition, predicting different stresses or deformations for an identical input loading.

### 3.2 Smoothness condition

It is observed in experiments that "the stress rate induced by the identical strain rate changes continuously for a continuous change of stress state". This property is called the smoothness condition and is expressed mathematically as follows:

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{O}} \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, H_i) = \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma}, H_i) \quad (12)$$

A smooth response of stress-strain relation is not described in constitutive equations violating the smoothness condition, causing discontinuous change of tangent modulus for the elastoplastic constitutive equations assuming the yield surface enclosing a purely-elastic domain where the tangent modulus changes abruptly from the elastic to the elastoplastic state.

The rate-linear constitutive equation is described as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{K}^{ep}(\boldsymbol{\sigma}, H_i)\mathbf{:d} \quad (13)$$

where the fourth-order tensor $\mathbf{K}^{ep}$ is the elastoplastic modulus, which is a function of the stress and internal variables, can be described generally as

$$\mathbf{K}^{ep} = \frac{\partial\overset{\circ}{\boldsymbol{\sigma}}}{\partial\mathbf{d}} \quad (14)$$

Consequently, Eq. (12) can be rewritten as

$$\lim_{\delta\boldsymbol{\sigma}\to\mathbf{O}} \overset{\circ}{\boldsymbol{\sigma}}(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, H_i) = \mathbf{K}^{ep}(\boldsymbol{\sigma}, H_i) \quad (15)$$

## 4. Subloading surface model

The natural postulate "the plastic strain rate develops gradually as the stress approaches the yield surface" is introduced in the subloading surface model. It is necessary to adopt an appropriate measure describing the degree of approach to the yield state. Then, let the subloading surface, be introduced, which always passes through the current stress point and has similar shape and orientation to the yield surface, while the yield surface is renamed the normal-yield surface (Hashiguchi, 1980, 1989, 2013; Hashiguchi et al., 2012). The subloading surface is described by

$$f(\hat{\boldsymbol{\sigma}}) = RF(H) \quad (16)$$

*R* is the ratio of the size of the subloading surface to that of the normal-yield surface and it is called the normal-yield ratio.

The time-derivative of Eq. (16) leads to

$$\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial\boldsymbol{\sigma}}\mathbf{:}\overset{\circ}{\boldsymbol{\sigma}} - \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial\boldsymbol{\sigma}}\mathbf{:}\overset{\circ}{\boldsymbol{\alpha}} - \dot{R}F - R\dot{F} = 0 \quad (17)$$

We adopt the associated flow rule

$$\mathbf{d}^p = \dot{\lambda}\hat{\mathbf{N}} \quad (\dot{\lambda} \geq 0) \quad (18)$$

where $\dot{\lambda}$ is the plastic multiplier, i.e. the positive proportionality factor designating the magnitude of plastic strain rate and

$$\hat{\mathbf{N}} \equiv \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \Bigg/ \left\| \frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\hat{\mathbf{N}}\|=1) \tag{19}$$

The evolution rule of the normal-yield ratio is given by

$$\dot{R} = U(R)\,\|\mathbf{d}^p\| \quad \text{for} \quad \mathbf{d}^p \neq \mathbf{O} \tag{20}$$

where the function $U$ of $R$ is given as

$$U(R) = u \cot\left(\frac{\pi}{2} \frac{\langle R - R_e \rangle}{1 - R_e}\right) \tag{21}$$

$u$ is the material constant.

Substituting Eq. (20) into Eq. (17), the magnitude of plastic strain rate and the plastic strain rate are given by

$$\dot{\lambda} = \frac{\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}}{M^p}, \quad \mathbf{d}^p = \frac{\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}}{M^p} \hat{\mathbf{N}} \tag{22}$$

where

$$M^p \equiv \left(\frac{F'}{F} h + \frac{U}{R}\right) \hat{\mathbf{N}} : \boldsymbol{\sigma} \tag{23}$$

Substituting Eq. (2) and (22) into Eq. (1), the strain rate is given by

$$\mathbf{d} = \mathbf{E}^{-1} : \overset{\circ}{\boldsymbol{\sigma}} + \frac{\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}}{M^p} \hat{\mathbf{N}} \tag{24}$$

from which we have

$$\dot{\lambda} = \frac{\hat{\mathbf{N}} : \mathbf{E} : \mathbf{d}}{M^p + \hat{\mathbf{N}} : \mathbf{E} : \hat{\mathbf{N}}} \tag{25}$$

The stress rate is described from Eqs. (24) and (25) as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E} : \mathbf{d} - \left\langle \frac{\hat{\mathbf{N}} : \mathbf{E} : \mathbf{d}}{M^p + \hat{\mathbf{N}} : \mathbf{E} : \hat{\mathbf{N}}} \right\rangle \mathbf{E} : \hat{\mathbf{N}} \tag{26}$$

The plastic strain rate is induced even in the subyield surface, depending on the normal-yield ratio and the direction of strain rate, while the stress lies always on the subloading surface playing the role of loading surface. Therefore, a judgment of whether or not the yield condition is satisfied is not required. Then, the loading criterion is given by

$$\left.\begin{array}{ll} \mathbf{d}^p \neq \mathbf{O}: & \hat{\mathbf{N}} : \mathbf{E} : \mathbf{d} > 0 \\ \mathbf{d}^p = \mathbf{O}: & \text{otherwise} \end{array}\right\} \tag{27}$$

## 5. Incorporation of tangential inelastic strain rate

As seen in Eq. (22), the inelastic strain rate in the traditional constitutive equation has the limitations: The inelastic strain rate depends solely on the stress rate component normal to the yield surface, called the normal stress rate, but it is independent of the component tangential to the yield surface, called the tangential stress rate. On the other hand, it has been verified by experiments that an inelastic strain rate induced by the deviatoric part of the tangential stress rate, called the deviatoric tangential stress rate, influences considerably on a deformation in the non-proportional loading process deviating from the proportional loading path normal to the yield surface. It is called the tangential inelastic strain rate. The subloading surface model has been extended to describe the deviatoric tangential stress rate (Hashiguchi and Tsutsumi, 2001, 2003; Tsutsumi and Hashiguchi, 2005).

First, assume that the strain rate is decomposed additively into elastic and plastic strain rates and further the tangential-inelastic strain rate $\mathbf{d}^t$ as follows:

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p + \mathbf{d}^t \tag{28}$$

where $\mathbf{d}^t$ is induced by the deviatoric tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}'$ which is decomposed into the deviatoric normal stress rate $\overset{\circ}{\boldsymbol{\sigma}}'_n$ and the deviatoric tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}'_t$:

$$\overset{\circ}{\boldsymbol{\sigma}}' = \overset{\circ}{\boldsymbol{\sigma}}'_n + \overset{\circ}{\boldsymbol{\sigma}}'_t \tag{29}$$

where

$$\overset{\circ}{\boldsymbol{\sigma}}' = \mathcal{I}' : \overset{\circ}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}} - \frac{1}{3}(\mathrm{tr}\overset{\circ}{\boldsymbol{\sigma}})\mathbf{I} \tag{30}$$

$$\left.\begin{array}{l} \overset{\circ}{\boldsymbol{\sigma}}'_n \equiv (\hat{\mathbf{n}}' \otimes \hat{\mathbf{n}}') : \overset{\circ}{\boldsymbol{\sigma}} = (\hat{\mathbf{n}}' : \overset{\circ}{\boldsymbol{\sigma}})\,\hat{\mathbf{n}}' : \overset{\circ}{\boldsymbol{\sigma}} \\ \overset{\circ}{\boldsymbol{\sigma}}'_t \equiv \hat{\mathfrak{T}}' : \overset{\circ}{\boldsymbol{\sigma}} = \overset{\circ}{\boldsymbol{\sigma}}' - \overset{\circ}{\boldsymbol{\sigma}}'_n \;\; (\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}'_t = 0) \end{array}\right\} \tag{31}$$

$$\begin{aligned} \hat{\mathbf{n}}' &\equiv \left(\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}}\right)' \Bigg/ \left\| \left(\frac{\partial f(\hat{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}}\right)' \right\| \\ &= \frac{\hat{\mathbf{N}}'}{\|\hat{\mathbf{N}}'\|} (\neq \hat{\mathbf{N}}') \;\; (\|\hat{\mathbf{n}}'\| = 1, \;\; \|\hat{\mathbf{N}}'\| \neq 1) \end{aligned} \tag{32}$$

$$\hat{\mathfrak{T}}' \equiv \mathcal{I}' - \hat{\mathbf{n}}' \otimes \hat{\mathbf{n}}', \quad \hat{\mathfrak{T}}'_{ijkl} \equiv \mathcal{I}'_{ijkl} - \hat{n}'_{ij}\hat{n}'_{kl} \tag{33}$$

Hereinafter, the deviatoric-tangential tensor is denoted as $(\;)'_t$, i.e. $\mathbf{t}'_t \equiv \hat{\mathfrak{T}}' : \mathbf{t}$ for arbitrary second-order tensor $\mathbf{t}$, exploiting the *fourth-order deviatoric tangential projection tensor* $\hat{\mathfrak{T}}'$.

Now, assume that the tangential inelastic strain rate $\mathbf{d}^t$ is related linearly to the tangential deviatoric stress rate $\overset{\circ}{\boldsymbol{\sigma}}'_t$.

$$\mathbf{d}^t = \frac{T}{2G} \overset{\circ}{\boldsymbol{\sigma}}'_t \tag{34}$$

where $T$ is a monotonically-increasing function of $R$ in addition to the stress and the internal variable, i.e.

$$T = \xi R^{\tau} \tag{35}$$

$\tau(\geq 1)$ is the material constant and $\xi$ is the function of the stress $\boldsymbol{\sigma}$ and the internal variables $H$ as

$$\xi = \xi(\boldsymbol{\sigma}, H) \tag{36}$$

in general. Then, the tangential inelastic strain rate develops as the stress approaches the yield surface by virtue of the advantage of the subloading surface model.

Substituting Eqs. (24) and (34) into Eq. (28), the strain rate is given by

$$\begin{aligned} \mathbf{d} &= \mathbf{E}^{-1} : \overset{\circ}{\boldsymbol{\sigma}} + \frac{\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}}{M^p} \hat{\mathbf{N}} + \frac{T}{2G} \overset{\circ}{\boldsymbol{\sigma}}'_t \\ &= \left(\mathbf{E}^{-1} + \frac{\hat{\mathbf{N}} \otimes \hat{\mathbf{N}}}{M^p} + \frac{T}{2G} \hat{\mathfrak{T}}'\right) : \overset{\circ}{\boldsymbol{\sigma}} \end{aligned} \tag{37}$$

In what follows, let the inverse expression of Eq. (37) be derived, provided that the elastic tangent modulus tensor $\boldsymbol{E}$ is given by Eq. (3) in the Hooke's type. Therefore, note that $\dot{\lambda}$ is given by Eq. (25) itself because of $\hat{\mathbf{N}} : \mathbf{E} : \overset{\circ}{\boldsymbol{\sigma}}'_t = 2G\hat{\mathbf{N}} : \overset{\circ}{\boldsymbol{\sigma}}'_t = 0$, noting Eq. $(31)_2$.

Then, it follows from Eq. (37) with Eq. (3) that

$$\mathbf{d}' = \frac{1}{2G}\overset{\circ}{\boldsymbol{\sigma}}' + \frac{\hat{\mathbf{N}}:\overset{\circ}{\boldsymbol{\sigma}}}{M^p}\hat{\mathbf{N}}' + \frac{T}{2G}\overset{\circ}{\boldsymbol{\sigma}}'_t \tag{38}$$

from which one has the following relation, considering $\hat{\mathbf{N}}'_t \equiv \hat{\boldsymbol{\mathfrak{T}}}': \mathbf{N} = \hat{\mathbf{N}}' - (\hat{\mathbf{n}}':\hat{\mathbf{N}})\hat{\mathbf{n}}' = \mathbf{O}$,

$$\mathbf{d}'_t = \frac{1}{2G}(1+T)\,\overset{\circ}{\boldsymbol{\sigma}}'_t \tag{39}$$

where

$$\mathbf{d}'_t \equiv \hat{\boldsymbol{\mathfrak{T}}}': \mathbf{d} = \mathbf{d}' - (\hat{\mathbf{n}}':\hat{\mathbf{N}})\hat{\mathbf{n}}' \,. \tag{40}$$

Substituting further Eq. (39) into Eq. (34), the tangential inelastic strain rate is given by

$$\mathbf{d}^t \equiv \frac{T}{1+T}\mathbf{d}'_t \,. \tag{41}$$

The stress rate is derived from Eqs. (2), (21), (25) and (41) as follows:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}:\mathbf{d} - \frac{\langle \hat{\mathbf{N}}:\mathbf{E}:\mathbf{d}\rangle}{M^p + \hat{\mathbf{N}}:\mathbf{E}:\hat{\mathbf{N}}}\mathbf{E}:\hat{\mathbf{N}} - \frac{2GT}{1+T}\mathbf{d}'_t \tag{42}$$

i.e.

$$\overset{\circ}{\boldsymbol{\sigma}} = \left(\mathbf{E} - \frac{\mathbf{E}:\hat{\mathbf{N}}\otimes\hat{\mathbf{N}}:\mathbf{E}}{M^p + \hat{\mathbf{N}}:\mathbf{E}:\hat{\mathbf{N}}} - \frac{2GT}{1+T}\hat{\boldsymbol{\mathfrak{T}}}'\right):\mathbf{d} \tag{43}$$

Eq. (37) or (43) has been applied to the predictions of the non-proportional loading and the plastic instability phenomena.

## 6. Concluding remarks

The physical and mathematical background of the hypoelastic-based plasticity is deliberated first in this article. Then, the constitutive equations based on the subloading surface concept are formulated within the framework of the hypoelastic-based plasticity and the applications to the descriptions of the wide classes of elastoplastic deformation are shown. The salient features of the concept and the constitutive equations based on this concept are summarized as follows:

(1) It is based on the quite natural concept that the plastic deformation develops as the stress approaches the yield surface and thus it possesses the high generality and the capability of describing accurately elastoplastic deformations of wide classes of materials.

(2) It fulfills the smoothness condition, describing always the smooth elastic-plastic transition, and possesses the automatic controlling operations to attract the stress to the yield surface and the plastic strain to the isotropic hardening stagnation surface simultaneously.

(3) It is capable of describing the finite deformation and rotation under an infinitesimal elastic deformation.

(4) It is capable of describing the monotonic and the cyclic loading behavior pertinently. In addition, the non-proportional loading behavior and the plastic instability phenomena can be described appropriately since it incorporates the tangential-inelastic strain rate induced by the rate of stress inside the yield surface, fulfilling always the continuity condition

(5) It possesses the distinctive advantage that the stress is automatically attracted to the yield surface. Therefore, it enables us to adopt rather large incremental steps in the forward Euler numerical calculation without the incorporation of a particular algorithm to pull back the stress to the yield surface. Further, the plastic strain is automatically attracted to the isotropic hardening stagnation surface for metals, for which it is difficult for the return-mapping projection to be exploited.

These advantages would be activated in large scale finite element analyses solving a big global stiffness matrix. Needless to say, infinitesimal increments must be input for the deformation analyses in the curved loading process and under a material rotation in numerical calculations not only by the forward-Euler method but also by the return-mapping scheme. Consequently, the physical and the mathematical pertinences and the numerical convenience are materialized in the subloading surface model.


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