Return Mapping Technique for the Extended Subloading Surface Model with Tangential Plasticity†

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Abstract

Most of the phenomenological elastoplastic models adopting a plastic flow rule with a single yield surface tend to predict an unrealistic material response especially when they are subjected to the non-proportional loading condition with a sudden change of the loading path. The work presented in this paper aims to propose an efficient numerical technique to correct the excessive stiffness of the conventional elastoplastic models by introducing the so-called tangential plasticity [1] and combining it with the return mapping technique for an accurate and faster computation [2].

KEY WORDS: (non-proportional loading), (unconventional plasticity), (cyclic plasticity), (vertex-effect), (return mapping).

1. Introduction

Phenomenological plasticity models are commonly formulated with the assumptions on the plastic flow (normality) rule that the inelastic stretching can only evolve along the normal direction to the plastic potential surface, and also its magnitude depends only on the stress rate component normal to the yield surface. That is, in the plastic loading case, the direction of inelastic stretching is uniquely defined regardless of that of the stress rate. Furthermore, the magnitude of the inelastic stretching is not affected by the stress rate component tangential to the yield surface. Although these assumptions might give satisfactory numerical results under a certain condition, such as monotonic loading within small deformation range, the deficiency has been pointed out especially when the non-proportional loading, accompanied with a sudden change of the loading direction, is subjected to the materials and structures. Therefore, the use of these models can lead to an underestimation of the inelastic deformation with a stiffer response against non-proportional loading, and also the plastic instability phenomena with strain localization tends to be overestimated [1].

To overcome these drawbacks, a large number of extensive works have been conducted to include the so-called vertex effect of a yield surface on the phenomenological plasticity models (i.e. [3]-[5]). Among them the tangential plasticity model together with the subloading surface concept has crucial advantage for the description of general deformation behavior under cyclic and non-proportional loading conditions, since it is categorized in the unconventional plasticity model and has a mathematical structure to describe a smooth elastic-plastic transition. The model can describe the dependence of not only the magnitude but also the direction of the inelastic stretching on the stress rate direction by considering the tangential plasticity (c.f. Hashiguchi and Tsutsumi [1]). The applicability of the model have been discussed for both analytical bifurcation problems (i.e. [6]-[10]) and general non-proportional loading behavior for soils (Tsutsumi and Hashiguchi [11], Tsutsumi and Kaneko [12]).

On the other hand, many practical engineering problems require high computational effort. This is a consequence either of the complexity of some physical phenomena and of the evolution of the computational powers, which let us pushing forward the limit of numerical calculations; especially in F.E. analyses with elevate number of degrees of freedom or special loading conditions. Therefore the recourse to numerical techniques has become frequent in order to save time, without giving up the accuracy of the solution.

The main purpose of this work is to combine the tangential plasticity effect for the extended subloading surface model with the efficiency of the return mapping algorithm ([2],[13]) as a useful tool in numerical computation. The key idea lays on the assumption that the tangential inelastic stretch doesn't affect the material hardening, allowing to compute separately the elastoplastic stress rate by means of the return mapping

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algorithm and subsequently the tangential inelastic term using a set of parameters which depend on the stress state. In the present paper the attention will be focused on metals constitutive equations with mix-hardening.

2. Constitutive equations and numerical procedure

Before dealing with the updated constitutive equations of the unconventional plasticity model it is useful to introduce the main aspects of the numerical technique itself. Return mapping algorithm has been firstly proposed by Wilkins [14], subsequently generalized by Simo and Ortiz [15], and then used in many forms by other authors (i.e. Krieg and Krieg [16], Phillinger [17], etc.). However the two main common formulations are: the closest projection method (CPPM) and the cutting plane algorithm (CPA), as reviewed by literatures (c.f. Hashiguchi [18]). The first is a complete implicit method which allows to compute all the variables of the system in order to satisfy the set of equations composed by the yield condition and the evolution law of the internal variables. This technique is based on the concept of the consistent linearization, introduced for the first time by Huges and Pister [19], which leads to an asymptotic quadratic rate of convergence to the solution, making the CPPM really attractive to use.

One problem that arises with this formulation is due to the difficulty on calculating the second order derivatives of the yield function respect to the stress state for setting up the local matrix. Especially when dealing with geomaterials, the plastic potential can have a complicated mathematical expression which makes particularly difficult to define the direction of the plastic flows [20].

On the other hand the cutting plane algorithm is an incomplete implicit method which results in the impossibility to compute analytically the consistent tangent modulus as in the previous formulation [13]. This represents a serious limitation since a quadratic convergent Newton-Raphson scheme cannot be used for the finite element global equilibrium. However the linearization of the consistency conditions allows to simplify the convergence procedure, especially considering that no second order derivatives has to be computed; moreover a numerical consistent tangent operator can be furnished adopting the strategy proposed by Miehe [21], supplying the lack of its analytical formulation.

It has to be underlined that the recourse to unconventional theories is necessary when dealing with cyclic plasticity due to the fact that they allow to take into account irreversible contributes even for small oscillations of the stress state in the neighborhood of the yield surface. The mechanism for such behavior can be different accordingly with the theory adopted (Multi-surface model [22] and [23], Infinite surface model [24], Two surface model [25], Single surface model [26]) but basically all of them abolish the distinction between elastic and plastic domains in order to generate inelastic strains once the stress increases again after a previous unload of the sample.

The subloading surface [27] is the only model among the previous ones which has the ability to catch a smooth elastoplastic transition by means of its solid and consistent mathematical implant. To achieve this result a new surface, named subloading surface, has been created inside the conventional plastic yield (here renamed normal-yield) through a similarity transformation, whose center is not fixed in the stress space but it moves following the plastic strain rate evolution.

2.1 Extended Subloading Surface Model

The main variables of the model will be now introduced without a complete explanation of the theory, referring the reader to [1], [2] and [28] for a more detailed discussion. Observing Figure 1 the normal-yield and subloading surface equations can be formulated as follows:

$$f(\hat{\boldsymbol{\sigma}}) = F(H), \quad \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\alpha}$$
$$f(\bar{\boldsymbol{\sigma}}) = RF(H) \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress, $\boldsymbol{\alpha}$ is the so called back-stress, H the isotropic hardening variable.

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}}, \quad \bar{\boldsymbol{\alpha}} = \mathbf{s} - R\hat{\mathbf{s}},$$
$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \mathbf{s}, \quad \hat{\mathbf{s}} = \mathbf{s} - \boldsymbol{\alpha} \tag{2}$$

The variables introduced in Eq. (2) are respectively: the stress observed from the conjugate back-stress; the conjugate-back stress itself (function of the similarity ratio); the stress observed from the similarity center and, in the end, the similarity center seen from the back-stress.

According to Rudnick and Rice [29], and subsequently remarked by Hashiguchi and Tsutsumi [1], only the deviatoric part of the tangential stress rate influences the inelastic deformation. This crucial evidence, together with the assumption that the tangential effect doesn't affect the hardening behavior [28], allows to split the computation of the inelastic normal and tangential components of the inelastic stretch. The former can be evaluated by means of the return mapping and the latter by a simple formulation derived through some mathematical passages. In the following paragraphs the two terms will be presented separately, but the reader should keep in mind that they both contribute additively in the formation of the total deformation as stressed out by Eq.(10). It has to be preliminary said that the strategy presented in this paper has general validity but the set of equations presented in Eq.(4), (5) and (18) hold just for metals with a Von Mises yield surface.

The starting point for the return mapping formulation is that to freeze all the plastic variables (i.e. isotropic hardening variable, back-stress, yield surface) between a generic n step to the subsequent $n+1$, imaging that material behavior is perfectly linear elastic and performing the so called *trial state*. It should be underlined that this hypothesis may not, and in general it will not, be an admissible state for the material because in most of the cases it will overestimate the stress level. A first check is here necessary to verify if an irreversible response has been activated or if the elastic assumption is effectively the correct answer for the fulfilment of equilibrium. This statement can be analytically formulated by the following inequality:

$$\begin{aligned} &\mathrm{f}(\boldsymbol{\sigma}_{n+1}^{(trial)}) \le R_n F(H_n),\ \boldsymbol{\varepsilon}_{n+1}^{p} = 0,\ \boldsymbol{\sigma}_{n+1}^{(trial)} = \boldsymbol{\sigma}_{n+1}^{(final)} \\ &\mathrm{f}(\boldsymbol{\sigma}_{n+1}^{(trial)}) > R_n F(H_n),\ \boldsymbol{\varepsilon}_{n+1}^{p} \ne 0,\ \boldsymbol{\sigma}_{n+1}^{(trial)} \ne \boldsymbol{\sigma}_{n+1}^{(final)} \end{aligned} \quad (3)$$

If $\boldsymbol{\sigma}_{n+1}^{(trial)}$ satisfy the second condition of Eq. (3) the plastic unknowns must be updated through an internal iterative procedure (Eq. (4)) inside the step, which starts with the linearization of the consistency condition (Eq. (1)) and leads to the computation of the k+1 proportionally factor (Eq. (5)) as a function of all the internal variables expressed at the k sub-step.

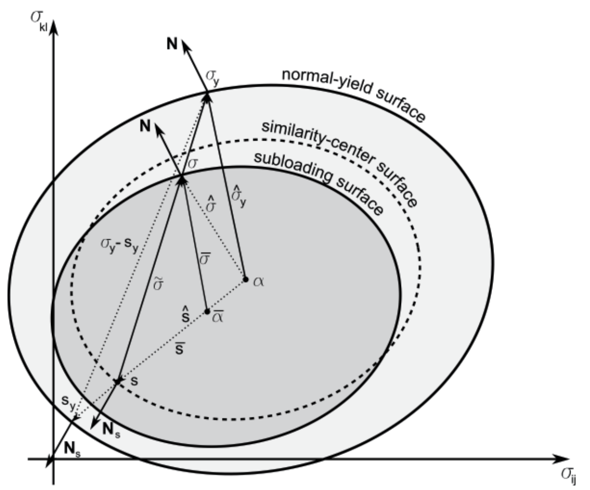

Figure 1 - Extended subloading surface model.

$$\mathrm{f}(\bar{\boldsymbol{\sigma}}_{n+1}^{(k)}) = \sqrt{\frac{3}{2}}\left|\bar{\boldsymbol{\sigma}}'^{(k)}_{n+1}\right| \qquad U(R) = u\cot\left(\frac{R_{n+1}^{(k)} - R_e}{1 - R_e}\right)$$

$$\mathbf{N} = \hat{\mathbf{N}} = \bar{\mathbf{N}} = \bar{\mathbf{N}}' = \frac{\bar{\boldsymbol{\sigma}}'^{(k)}_{n+1}}{\left|\bar{\boldsymbol{\sigma}}'^{(k)}_{n+1}\right|} \qquad h = \sqrt{\frac{3}{2}}$$

$$M_p = \bar{M}_p = \mathbf{N}_{n+1}^{(k)} : \left[\sqrt{\frac{3}{2}}\frac{F'^{(k)}_{n+1}}{F^{(k)}_{n+1}}\hat{\boldsymbol{\sigma}}_{n+1}^{(k)} + \mathbf{a}_{n+1}^{(k)} + \frac{U_{n+1}^{(k)}}{R_{n+1}^{(k)}}\tilde{\boldsymbol{\sigma}}_{n+1}^{(k)} \right. \quad (4)$$

$$\left. + c\left(1 - R_{n+1}^{(k)}\right)\left\{\frac{\bar{\boldsymbol{\sigma}}_{n+1}^{(k)}}{R_{n+1}^{(k)}} - \frac{\hat{\mathbf{s}}_{n+1}^{(k)}}{\chi}\right\}\right]$$

$$\mathring{\mathbf{s}} = \left|\boldsymbol{\varepsilon}^p\right|\left[c\left\{\frac{\bar{\boldsymbol{\sigma}}_{n+1}^{(k)}}{R_{n+1}^{(k)}} - \frac{s_{n+1}^{(k)}}{\chi}\right\} + \mathbf{a}_{n+1}^{(k)} + \sqrt{\frac{2}{3}}\frac{F'^{(k)}_{n+1}}{F^{(k)}_{n+1}}\hat{\mathbf{s}}_{n+1}^{(k)}\right]$$

$$\lambda_{n+1}^{(k+1)} = \frac{\left(\mathbf{N}_{n+1}^{(k)} : \mathring{\boldsymbol{\sigma}}_{n+1}^{(k)}\right)}{f(\boldsymbol{\sigma}_{n+1}^{(k)})}\left[\frac{\mathrm{f}(\boldsymbol{\sigma}_{n+1}^{(k)}) - R_{n+1}^{(k)}F_{n+1}^{(k)}}{M^p + \mathbf{N}_{n+1}^{(k)} : \mathbf{E} : \mathbf{N}_{n+1}^{(k)}}\right] \quad (5)$$

Using the additive decomposition of the strain it is possible to correct the trial stress state to consider the nonlinear behavior of the material:

$$\boldsymbol{\sigma}_{n+1}^{(k+1)} = \boldsymbol{\sigma}_{n+1}^{(k)} + d\boldsymbol{\sigma}_{n+1}^{p(k+1)} = \boldsymbol{\sigma}_{n+1}^{(k)} - \mathbf{E} : \lambda_{n+1}^{(k+1)}\mathbf{N}_{n+1}^{(k)} \quad (6)$$

At this point a new sub k-iteration can be performed from Eq. (4) until the fulfilment of the condition $f(\bar{\boldsymbol{\sigma}}_{n+1}^{(k+1)}) = R_{n+1}^{(k+1)}F_{n+1}^{(k+1)}$, meaning that the stress has finally reached the correct plastic surface (this can be numerically translated by imposing a tolerance under which the convergence can be considered satisfied, i.e. residual function $\left(f(\bar{\boldsymbol{\sigma}}_{n+1}^{(k+1)}) - R_{n+1}^{(k+1)}F_{n+1}^{(k+1)}\right) / F_{n+1}^{(k+1)} \le Toll$).

The set of equations presented so far are responsible for the local convergence on a single quadrature point but they cannot satisfied the global equilibrium for all the body, which will be in general not fulfilled.

The displacements (and consequently deformations) must be corrected through a global iterative procedure which solves once again the system $\mathbf{K}\,\mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{F}$, where $\mathrm{d}\mathbf{F}$ is the residual vector computed as the difference between external loads and the assembly vector of the internal forces at each Gauss point. The $\mathrm{d}\boldsymbol{\varepsilon}$ vector, derived from $\mathrm{d}\mathbf{u}$, is used as an input for a new local convergence using return mapping algorithm and the whole procedure should end when local and global equilibrium are satisfied.

Figure 2 represents the convergence of the stress on the plastic surface for a general quadrature point, where the linearization of the consistency condition can be seen as the plane where each *k*-sub iteration converges.

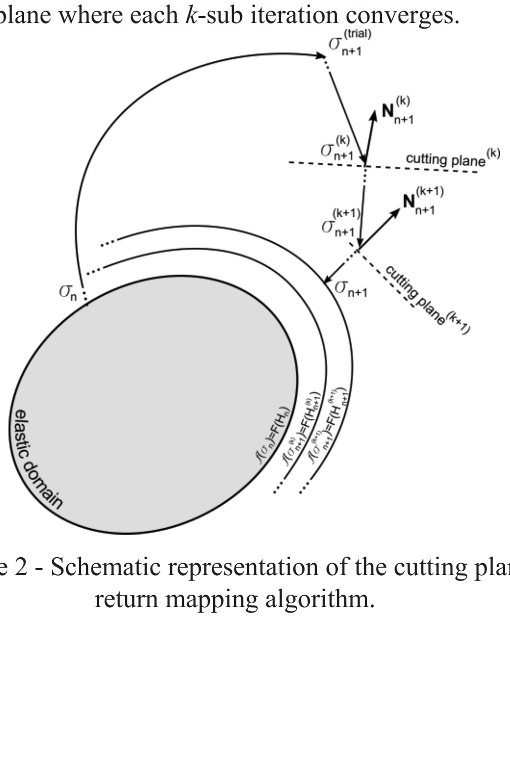

Figure 2 - Schematic representation of the cutting plane return mapping algorithm.

2.2 Extended to include the tangential plasticity

Once estimated the normal component of the stress rate, it is possible to compute the tangential one (Eq. (7)-(8)) considering it as linearly related to the tangential stretch (Eq.(9)):

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma} - \frac{1}{3}\sigma_{ii}\mathbf{I} \quad (7)$$

$$\overset{\circ}{\boldsymbol{\sigma}}{}^* = \overset{\circ}{\boldsymbol{\sigma}}{}^*_n + \overset{\circ}{\boldsymbol{\sigma}}{}^*_t$$

$$\overset{\circ}{\boldsymbol{\sigma}}{}^*_n = (\mathbf{N} \otimes \mathbf{N})\overset{\circ}{\boldsymbol{\sigma}}{}^* \quad (8)$$

$$\overset{\circ}{\boldsymbol{\sigma}}{}^*_t = \overset{\circ}{\boldsymbol{\sigma}}{}^* - \overset{\circ}{\boldsymbol{\sigma}}{}^*_n$$

$$\mathbf{D}^t = A\overset{\circ}{\boldsymbol{\sigma}}{}^*_t \quad (9)$$

where A (≥ 0) is a generic scalar function that will be defined later.

The additive decomposition of the strain rate, assumed valid in the Eq.(6), is enriched adding a third component: elastic, plastic (toward the normal direction), and tangential inelastic:

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p + \mathbf{D}^t \quad (10)$$

Eq. (10) can be rewritten as a function of the corotational stress rate as follows:

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{tr(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M_p}\mathbf{N} + A\overset{\circ}{\boldsymbol{\sigma}}{}^*_t \quad (11)$$

Through some mathematical passages [1] it is possible to write the inverse relationship expressing the corotational stress rate as a function of the strain rate. In the following, focused on a metal plasticity modelling, some terms have been neglected as a consequence of considering plastically incompressibility of the material. Since metals are invariant to any change of hydrostatic pressure, the original formulation is further simplified to include just one normal vector (i.e. Eq.(4)):

$$\overset{\circ}{\boldsymbol{\sigma}} = \frac{1}{(1+2GA)}\left[\mathbf{E} - \frac{\mathbf{EN}\otimes\mathbf{EN}}{M_p + tr\mathbf{NEN}} + \right.$$
$$\left. + \frac{2GA}{3}\mathbf{E}\otimes\mathbf{I} + 2GA\frac{\mathbf{EN}\otimes\mathbf{N}}{M_p + tr\mathbf{NEN}}M_p\right]\mathbf{D} \quad (12)$$

At this point a specific form for the function A has been chosen to separate the elastoplastic part from the tangential one, in detail assuming:

$$A = \left(\frac{T}{2G - 2GT}\right) \quad (13)$$

it is possible to write:

$$\overset{\circ}{\boldsymbol{\sigma}} = \left\{\mathbf{E} - \frac{\mathbf{EN}\otimes\mathbf{EN}}{M_p + tr\mathbf{NEN}} - \overset{\circ}{\boldsymbol{\sigma}}{}^*_t\right\}\mathbf{D} \quad (14)$$

$$\overset{\circ}{\boldsymbol{\sigma}}{}^*_t = T\left(\mathbf{E} - \frac{\mathbf{EN}\otimes\mathbf{EN}}{M_p + tr\mathbf{NEN}} - \frac{1}{3}\mathbf{E}\otimes\mathbf{I}\right.$$
$$\left. - \frac{\mathbf{EN}\otimes\mathbf{N}}{M_p + tr\mathbf{NEN}}M_p\right)\mathbf{D} \quad (15)$$

where T is a variable which depends on the stress state (by means of the similarity ratio) and on two material parameters ($0 \leq \xi \leq 1$, $b \leq 0$), in the same way as presented in [1]:

$$T = \xi R^b \quad (16)$$

Looking at the right side of Eq.(14) it is easy to see that the first two addends represent the elastoplastic part of the stress rate, thus all the terms in the square brackets can be regarded as the deviatoric tangential stress rate part. This contribute is then subtracted from the total stress, obtained from Eq.(6), once local convergence in return mapping is fulfilled

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}^{(k)}_{n+1} - \mathbf{E} : \lambda^{(k+1)}_{n+1}\mathbf{N}^{(k)}_{n+1} - \overset{\circ}{\boldsymbol{\sigma}}{}^*_t \quad (17)$$

The problem to compute the stress in two separate steps lays on the fact that, when the elastoplastic part satisfies the consistency condition (Eq.(3)), a unique set of values for the similarity center R and the size of the yield surface F is possible. Therefore if the stress is 'relaxed', a deviation along a direction tangential to the plastic potential at the current stress is performed, bringing the point to lay outside the subloading surface with loss of local equilibrium. Moreover this aspect is enhanced whenever a large step simulation is carried out, since the entity of the deviation is directly proportional to the magnitude of the deviatoric tangential stress rate.

A simple solution adopted in this paper is the one of performing a correction of the stress state by means of a sort of 'single step return mapping operation'. The procedure can be better understood having a look at Figure 3: once the trial stress has been brought back to the correct plastic surface the tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}{}^*_t$ is applied generating the point $\boldsymbol{\sigma}^{(k+1),2}_{n+1}$.

In order to satisfy the global equilibrium and the consistency condition at the same time, it is necessary to bring back the stress to the F_{n+1} or $R_{n+1}F_{n+1}$ (in case R is less than unity) surface by means of a vector directed towards the center of the surface itself (i.e. the back stress or the conjugate back stress) and which magnitude can be easily computed using the first of Eq.(4). In formulas:

$$
\begin{aligned}
&1)\ \boldsymbol{\sigma}_{n+1}^{(k+1),2} = \boldsymbol{\sigma}_{n+1}^{(k+1),1} - \overset{\circ}{\boldsymbol{\sigma}}_t^{*} \\
&2)\ \mathbf{N}^{,2} = \partial f(\overline{\boldsymbol{\sigma}}_{n+1}^{(k+1),2}) \Big/ \left|\partial f(\overline{\boldsymbol{\sigma}}_{n+1}^{(k+1),2})\right| \\
&3)\ \lambda_{n+1}^{\text{tangential}} = \sqrt{3/2}\left|\overline{\boldsymbol{\sigma}}_{n+1}^{(k+1),2}\right| - \sqrt{3/2}\left|\overline{\boldsymbol{\sigma}}_{n+1}^{(k+1),1}\right| \\
&4)\ \boldsymbol{\sigma}_{n+1}^{(k+1),\text{final}} = \boldsymbol{\sigma}_{n+1}^{(k+1),2} - \lambda_{n+1}^{\text{tangential}} \mathbf{N}^{,2}
\end{aligned} \tag{18}
$$

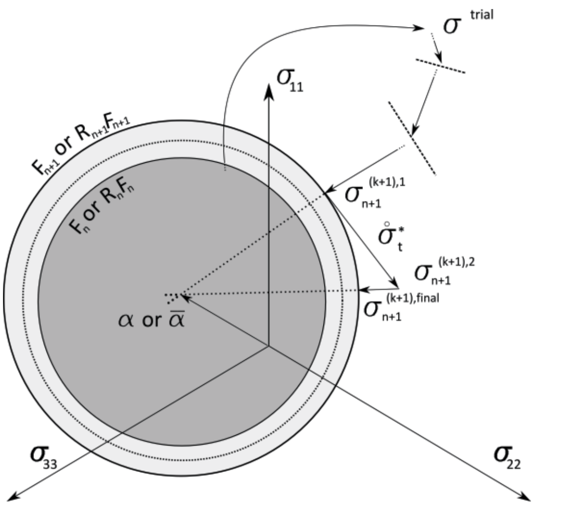

Figure 3 – correction method for the tangential stress rate contribute.

3. Concluding Remarks

The present work aimed to describe the inelastic stretching of materials subjected to non-proportional loading. Traditional elastoplastic models fail in predicting the correct amount of irreversible deformation when come to loading path with not-negligible stress rate component tangential to the plastic surface because of a purely associative formulation of the flow rule.

The system of equations formed by Eqs.(4) –(6) and (15) allows to compute the corotational stress rate in a completely innovative way uncoupling the effect of plastic strain directed along the normal to the yield surface and the one along its tangent. This strategy is based on the hypothesis that material hardening is not influenced by a possible inelastic contribute generated by tangential plasticity and thus the separation of the two terms is possible.

Moreover the assumption of linear dependency between the tangential stretch and the deviatoric tangential corotational stress rate appears to be very convenient for the implementations in F.E. simulations compared to more complex theories.

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