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抄録

壓縮を受ける矩形薄板の摂屈と破壊

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嘱託 工学士 近藤 一夫

本論文は矩形平板が其の双の邊に並行な圧縮荷重を受けて摂屈を起した後の状態と其破壊の条件を論じたものである。

薄板の平衡条件を以てる 6 個の微分方程式から出発して、摂屈による板の変形を定めた。但し次の様な四つの仮定を行って居る。

（1）

\[ T_1 = \frac{\partial T_2}{\partial y} = S_1 = S_2 = 0 \]

兹に \( T_1, T_2 \) は軸面の単位長に働く \( x, y \) 方向の直接応力、\( S_1, S_2 \) は同じく軸面単位長に働く剪断力を表す。但し荷重は \( y \) 軸に並行に働くものとする。

（2）圧縮荷重を受ける辺の変位は一定である。

（3）\( w \) の導函数について二次の項は無視し得る。

（4）荷重に並行な \( y \) 方向の波形は正弦波である。

斯くして、問題は結局次の微分方程式を解いて、\( x \) 方向の波形を以てる函数 \( \psi(x) \) を求める事に歸せられる。

\[ \frac{d^4 \psi}{dx^4} - 2 \frac{d^2 \psi}{dx^2} + \frac{\partial}{\partial \psi} \left( \frac{h}{ed} \right)^2 \left( \frac{e - \psi - \psi^3}{e_0} \right) = 0 \]

兹に \( \theta = \frac{\pi a x}{2b} \)，\( z \) \( b, 2d \) は夫々板の長さ及幅、\( \frac{n}{2} \) \( (y \) 方向の波数、\( e \) は見掛け上一様な \( y \) 方向の歪、\( e_0 \) は摂屈の臨界點に於ける支持された長方形の \( e \) を表す。
四辺を支持された方形板、又は\( y \)方向に無限に長い帯状板の場合については
\[
\begin{align*}
\frac{w}{2Adv^2} &= \frac{Kx}{\pi d} \ln \frac{2y}{d} - \sin \frac{\pi y}{2d}
\end{align*}
\]
と置いて此方程式の近似解を求め得る。之から導出される主要な結果は次の如くである。

(i) 此波形を以て応力の分布と、変形と共に荷重の増加する状態を計算してある。一様変の条件を飽く変因すれば、板の横断面の中央部に張力が起る事となる。實際には一様変の条件が必ずしも満足されて

(ii) 板の屈曲による表面の屈曲応力は直接圧縮のみによる両端の応力に匹敵する大きさである。従って板の破壊は圧縮と屈曲の合併応力が最大の点を始め、斯かる点は一般に板の両端に近いが真より少し内側にある。故に板の破壊を論ずには、圧縮した板の波形を出来るだけ正確に求める必要がある。

(iii) 第8図によって直ちに任意の薄板に対する許容荷重を知る事が出来る。

は等の結果は正方形板及び帯状板のみならず縦横比の相当大きな矩形板\((b > 2d)\)にも近似的に適用出来る。
Corrigenda to the Report No. 119.

(M. Yamamoto, K. Kondo, Buckling and Failure of Thin Rectangular Plates in Compression.)

<table>
<thead>
<tr>
<th>Page</th>
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<td>[ \frac{2\theta}{\pi} ]</td>
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<td>19</td>
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<td>Therefore .... alone.</td>
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<td>6</td>
<td>cannot have rely as great as</td>
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<td>21</td>
<td>29</td>
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</table>

* Instead of "Therefore .... alone," 1, 6-17, p. 18, put there, the following:

Therefore the maximum of the principal stress at the loop of the wave in y-direction $f_{by}$ or $f_{by}+f_{ay}$ occurs when its first derivative is equal to zero, i.e. when

\[
\frac{d^2\phi}{d\sigma^2} - \phi = 0
\]

or

\[
\frac{d^2\phi}{d\sigma^2} - \sqrt{1-\sigma^2} \frac{d\phi}{d\sigma} = 0.
\]

After calculation we know that the maximum of $f_{by}+f_{ay}$ is always greater than that of $f_{by}$ in the interval $\frac{\sigma_0}{2} \leq \frac{\pi}{4}$; and it exceeds by far the stresses at the side edges where the plate elements are subjected to pure compression alone. Hence it will be the maximum of the quantity:

\[
P^* = \frac{f_{by}}{E\sigma_0} = \frac{1}{2} \sqrt{1-\sigma^2 - \frac{d^2\phi}{d\sigma^2}} \phi + \left( \frac{\sigma}{\sigma_0} - 1 + \frac{\phi}{\phi_0} \right)
\]

\[
= \frac{1}{2} \sqrt{1-\sigma^2} A \frac{2K^2}{\pi} \left( \frac{2K^2}{\pi} \right) \left\{ \phi \left( \frac{2K^2}{\pi} \right) \left( 1 + 4\sigma^2 - 6\sigma^2 \sin^2 \frac{2K^2}{\pi} \right) + 1 \right\}
\]

\[
+ \frac{\sigma}{\sigma_0} - 1 + \frac{\phi}{\phi_0}
\]

which will predominate in determining the collapse point of the plate.

† Fig. 7, p. 19 is unnecessary. Instead of it, put there the following table (Table III):

<table>
<thead>
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<tr>
<td>45°</td>
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<td>45°</td>
</tr>
</tbody>
</table>

† The last figure (Fig. 8) is erroneous. The correct one is the following:

![Fig. 7](image-url)
No. 119.
(Published April, 1935.)

Buckling and Failure of Thin Rectangular Plates in Compression.

By
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and

Kazuo Kondo, Kōgakusi.

A number of investigators have worked on the problems of elastic stability, which have played a very important rôle in the recent progress of aeroplane constructions. Most of these studies, from the classical papers of Euler and Bryan down to the most elaborate ones of recent years, are concerned, generally, with the critical points where the elastic bodies in question enter a state of instability, assuming wave forms as anticipated from the given configurations.

The theoretical study of the state after buckling first engaged the attention of investigators only a few years ago, although the most eminent supposition about the state of stability of elastic bodies after buckling was made by Dr. K. Sezawa about ten years ago\(^{(1)}\). Th. v.

Kármán(1), E. E. Sechler(2), and L. H. Donnell(3) in C.I.T. were the first to attack this problem, of the rectangular plate in compression. They tried to simplify the complicated problem with the notion of "Effective Breadth", but no remarkable progress was attained until 1933, when an entirely new method of investigation was initiated by H.L. Cox(4) of England. His principle consists of

1. Introduction of the new idea that the edges of the rectangular plate are subjected to uniform displacement in the direction of the load rather than uniform stress.

2. The making of a certain assumption with respect to the wave form that the cross-section (parallel to the loaded edges) is likely to assume after buckling, so that the strain energy method could be applied in solving problem.

The first of these assumptions seems to be correct. The chief objection in Cox's solution, if any, however appears to lie in the second. To us it seems necessary to put aside the unsatisfactory assumption about the wave form and consequently of the stress distribution of the cross section. Starting from the general equations of equilibrium of the plate elements, the present authors have tried to eliminate the unsatisfactory points connected with Cox's solution.

1. Equations of Equilibrium of the Plate.

Let

\[ 2b, 2d, \text{ and } 2h = \text{length, width, and breadth of the rectangular plate respectively.} \]


(2) E.E. Sechler: The Ultimate Strength of Thin Flat Sheets in Compression, C.I.T. Pub. No. 27.

(4) H.L. Cox: The Buckling of Thin Plates in Compression, R. & M. 1554.
P = compressive load,

E = Young's modulus of the material,

\( \sigma \) = Poisson's ratio of the material,

\( w \) = displacement of the plate elements in the direction perpendicular to the original plane of the plate.

Let us take \( Ox, Oy \) along the adjacent edges of the panel parallel to the width and length respectively, the origin of the co-ordinate lying at the corner of the plate, and let the plate be subjected to compressive load parallel to the axis \( Oy \). The equations of equilibrium of the plate element may then be written\(^{(1)}\)

\[
\begin{align*}
\frac{\partial T_1}{\partial x} - \frac{\partial S_2}{\partial y} - N_1 \frac{\partial^2 w}{\partial x^2} - N_2 \frac{\partial^2 w}{\partial x \partial y} &= 0, \\
\frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial y} - N_1 \frac{\partial^2 w}{\partial x \partial y} - N_2 \frac{\partial^2 w}{\partial x^2} &= 0, \\
\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + T_1 \frac{\partial^2 w}{\partial x^2} - S_2 \frac{\partial^2 w}{\partial x \partial y} + S_1 \frac{\partial^2 w}{\partial x \partial y} + T_2 \frac{\partial^2 w}{\partial y^2} &= 0,
\end{align*}
\]

\( (1) \)

\[
\begin{align*}
\frac{\partial H_1}{\partial x} - \frac{\partial G_2}{\partial y} + N_2 &= 0, \\
\frac{\partial G_1}{\partial x} + \frac{\partial H_2}{\partial y} - N_1 &= 0, \\
G_1 \frac{\partial^2 w}{\partial y \partial x} - G_2 \frac{\partial^2 w}{\partial x \partial y} + H_1 \frac{\partial^2 w}{\partial x^2} + H_2 \frac{\partial^2 w}{\partial y^2} + S_1 + S_2 &= 0.
\end{align*}
\]

\( (2) \)


The notations used are those of Love:

\( T_1, T_2 \) = Direct stress per unit length of the section perpendicular to the axis of \( x, y \),

\( S_1 \) = Shearing force in \( y \)-direction per unit length of the section perpendicular to \( x \),

\( S_2 \) = Shearing force in \( x \)-direction per unit length of the section perpendicular to \( y \),

\( N_1, N_2 \) = Shearing forces in \( z \)-direction per unit length of the section perpendicular to \( x \) and \( y \) respectively,

\( H_1 \) = Couple in \( x \)-direction per unit length of the section perpendicular to \( x \),

\( H_2 \) = Couple in \( y \)-directions per unit length of the section perpendicular to \( y \),

\( G_1 \) = Couple in \( y \)-direction per unit length of the section perpendicular to \( y \),

\( G_2 \) = Couple in \( x \)-direction per unit length of the section perpendicular to \( y \).

Fig. 2.
The components of couples $G_1$, $G_2$, $H_1$, $H_2$ are expressed in terms of second partial derivatives of $w$ as follows:

\[
G_1 = -D \left\{ \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} \right\}, \\
G_2 = -D \left\{ \frac{\partial^2 w}{\partial y^2} + \sigma \frac{\partial^2 w}{\partial x^2} \right\}, \\
H_1 = -H_2 = (1 - \sigma)D \frac{\partial^2 w}{\partial x \partial y}.
\]

Hence from the first two equations of (2) it is found that

\[
N_1 = -D \left\{ \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} \right\}, \\
N_2 = -D \left\{ \frac{\partial^4 w}{\partial y^4} + \frac{\partial^4 w}{\partial x^2 \partial y^2} \right\}.
\]

Since the compressive load is in the direction of $y$, we may assume that

\[
S_1 = S_2 = T_1 = 0.
\]

Now the first two equations of (1) and the last of (2) will approximately be satisfied if the squares or the products of the second derivatives of $w$ were neglected. If $T_2$ is assumed to be independent of $y$, the third equation of (1), by using (4) and (5), gives the differential equation of the plate

\[
D \left\{ \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right\} - T_2 \frac{\partial^2 w}{\partial y^2} = 0.
\]

Equation (6) is of the same type as the so-called Bryan's equation\(^{(1)}\), used so often in connection with the classical theory of elastic

instability. In the present equation, \( T_2 \), which varies with \( x \) as a function of the wave form of the plate, is not constant, while in Bryan's equation, it is constant and independent of \( x \) and \( y \).

If we admit Cox's assumption that the loading condition is such that the plate is subjected to uniform overall strain rather than uniform stress, we may write

\[
-T_2 = 2hE\left\{ e - \frac{1}{4b} \int_0^{2b} \left( \frac{\partial w}{\partial y} \right)^2 dy \right\}, \quad (7)
\]

where \( e \) is the uniform overall strain.

If the loaded edges are simply supported, the wave form along the \( y \)-direction may be expressed by a sine curve. We can therefore write

\[
w = \frac{2b \sqrt{\frac{e_0}{n\pi}}} \sin \frac{n\pi y}{2b},
\]

where \( e_0 = \frac{n^2}{3(1-\sigma^2)} \frac{h^2}{d^2} \) is the critical strain of buckling of the square plate, whose side is equal to \( d \); \( n \) the number of half waves in \( y \)-direction due to buckling, and \( \phi \) a function of \( x \) alone,

whence

\[
-T_2 = 2hE\left\{ e - \frac{1}{4} e_0 \phi(x) \right\}, \quad (7')
\]

and

\[
\frac{d^4 \phi}{d\theta^4} - 2 \frac{d^2 \phi}{d\theta^2} + \phi\left( \frac{b}{nd} \right)^2 \left\{ \frac{e}{e_0} \phi - \phi^3 \right\} = 0 \quad (6')
\]

where

\[
\theta = \frac{n\pi}{2b} x
\]

2. Solution for the Case When the Plate is Simply Supported at Its Four Edges.

It is a very difficult matter to find the solutions of \((6')\) that will satisfy given boundary conditions at the side edges. For the case
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when these edges are simply supported, however, we may safely assume that the half wave length along axis $Oy$ is equal to $2d$ provided $b$ is large compared with $d$ ($b \leq 2d$). We shall therefore take $nd = b$, in order to make the problem simpler. With such an assumption, can an approximate solution for the simplified equation (6')

$$\frac{d^4\phi}{d\theta^4} - 2\frac{d^2\phi}{d\theta^2} - (4\frac{e}{e_0} - 1)\phi + \phi^3 = 0$$

be obtained, as follows:

By putting

$$\phi(e) = A sn\frac{2K\theta}{\pi} dn\frac{2K\theta}{\pi}$$

where $\theta = \frac{\pi}{2d}x$ and $A$ is an arbitrary constant, the differential equation (6'') may then be written

$$Asn\frac{2K\theta}{\pi} dn\frac{2K\theta}{\pi}\left\{a_0 + a_1 dn^2\frac{2K\theta}{\pi} + a_2 dn^4\frac{2K\theta}{\pi}\right\} = 0$$

where

$$a_0 = (61 - 76 k^2 + 16 k^4)\left(\frac{2K}{\pi}\right)^4 - (104 - 8 k^2)\left(\frac{2K}{\pi}\right)^2 - (4\frac{e}{e_0} - 1)$$

$$a_1 = (-180 + 120 k^2)\left(\frac{2K}{\pi}\right)^4 + 12\left(\frac{2K}{\pi}\right)^2 + \frac{A^2}{k^2}$$

$$a_2 = 120\left(\frac{2K}{\pi}\right)^4 - \frac{A^2}{k^2}$$

Only two constants, $A$ and $k$, are arbitrary, whereas it is necessary to have three arbitrary constants to make the three coefficients, $a_0, a_1, a_2$ equal to zero. We must therefore find the condition that will make the first member of (9) approximately equal to zero.

(1) The notations used in this paper are those of "K. Hayashi: - Fünfstellige Funktionentafeln. Berlin, Verlag von Julius Springer, 1930, P. 126-155."
But we have
\[ 2\theta_0(q^2)\theta_0(2v, q^2) = \theta_0^2(v) + \theta_0(v) \theta_0(2v) \]
\[ \theta_0(q^2)\theta_0(2v, q^2) = \theta_0(2v)\theta_0(v) \]

Hence the numerator of the following expression,
\[ a_0\theta_0^2(\theta) + a_1 k'\theta_0^2(\theta) \theta_0^2(\theta) + a_2 k'^2 \theta_0^2(\theta) \]
\[ \theta_0^2(\theta) \]

which is obtained by transforming the expression in \{ \} of (9), becomes
\[ 4 a_0\theta_0^2(\theta)\theta_0^2(2\theta, q^2) - (2 a_0 - a_1 k') \theta_0^2(q^2) \theta_0^2(2\theta, q^2) \]
\[ + (a_2 k'^2 - a_0) \theta_0^2(\theta) \]

If
\[ a_2 k'^2 - a_0 = 0 \]
\[ 4 \theta_0^2(q^2) a_0 -(2 a_0 - a_1 k') \theta_0^2(q^2) = 0 \]

the above expression is reduced to \( 8 a_0 \theta_0^2(q^2) \theta_0^2(4\theta, q^2) \), which may be considered approximately equal to zero when \( q \) is not too large, so that (10) is none other than the required condition. Hence
\[ A^2 = k^2 \left\{ 120 \left( \frac{2K}{\pi} \right)^4 - \frac{1}{k'^2 a_0} \right\} \]
\[ \frac{e}{e_0} = \frac{1}{4} \left\{ 61 - 76 k^2 + 16 k^4 \right\} \left( \frac{2K}{\pi} \right)^4 - (10 - 8 k^2) \left( \frac{2K}{\pi} \right)^2 - a_0 + 1 \}

where
\[ a_0 = \frac{120 k^2 - 60 \left( \frac{2K}{\pi} \right)^4 + 12 \left( \frac{2K}{\pi} \right)^2}{k^2 - 2 \left( \frac{\theta_0^2(q^2)}{\theta_0^2(q^2)} - 1 \right) \frac{1}{k'}} \]

---

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In the special case that \( k \) is equal to zero, we have

\[
a_0 = 48, \quad e = e_0, \quad A^2 = 0.
\]

This gives the critical point of buckling that has been obtained so far by a number of authors. Equation (9) is completely satisfied in this case, for we have \( q = 0 \) and \( \partial_2(4\theta, q^8) = 0 \), which agrees with the results from the various different methods.

We have another special case in which the condition

\[
a_0 = a_1 = a_2 = 0
\]

is entirely satisfied, resulting in differential equation (9), the conditions of which are

\[
\frac{e}{e_0} = \frac{1}{4} \left\{ (61 - 76k^2 + 16k^4)\left(\frac{2K}{\pi}\right)^4 - (10 - 8k^2)\left(\frac{2K}{\pi}\right)^2 + 1 \right\},
\]

\[
\frac{A^2}{k^2} = (180 - 120k^2)\left(\frac{2K}{\pi}\right)^4 - 12\left(\frac{2K}{\pi}\right)^2 = 120\left(\frac{2K}{\pi}\right)^4.
\]

whence

\[
1 - 5\left(\frac{2K}{\pi}\right)^2 (1 - 2k^2) = 0
\]

or

\[-4 + 7.5k^2 + 3.28125k^4 + 2.109375k^6 + 2.65655k^8 + \ldots = 0.\]

Hence

\[k^2 = 0.4225,\]

\[a = \arcsin k = 40^\circ 32.5',\]

\[
\frac{2K}{\pi} = 1.141,\]

\[A = 9.26, \quad \frac{e}{e_0} = 11.51.\]

The numerical values of \( A \) and \( P \) for each \( \frac{e}{e_0} \), are calculated and shown in Table I.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \frac{e}{e_0} )</th>
<th>( A )</th>
<th>( \frac{1}{4dhEe_0} )</th>
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</tr>
</tbody>
</table>

The differential equation (6a/\( \alpha \)) is entirely satisfied for the cases, which are specified with the Gothic type.
3. Load and Stress Distribution.

Substituting the result of the last article
\[
w = 2 A \frac{dY}{\pi} e_0 \frac{2K\theta}{\pi} \frac{dn}{2K\theta} \sin \frac{\pi y}{2d}
\]
(12)
in (7'), we get
\[
- \frac{T_2}{2hEe_0} = \frac{e}{e_0} \frac{A^2}{4} \left( \frac{sn^2 2K\theta}{\pi} - k^2 sn^4 2K\theta \right),
\]
\[
\frac{P}{4dhEe_0} = \frac{e}{e_0} \frac{A^2}{4} \left( 1 + \frac{1 - 2k^2}{k^2} \left( 1 - E^* \right) \right),
\]
where \(E^*\) denotes the complete elliptic integral of the second kind.

The numerical relation between \(- \frac{T_2}{2hEe_0}, \frac{e}{e_0}\) and \(\theta\) are shown in Table II and Fig. 3; and that between \(\frac{P}{4dhEe_0}\) and \(\frac{e}{e_0}\) in Table I and Fig. 4. The manner in which the edge load increases with the strain of the plate agrees very well with Cox's approximate result by the strain energy method.

**Table II Load Distribution.**

<table>
<thead>
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<th>(\theta)</th>
<th>(\alpha)</th>
<th>15°</th>
<th>20°</th>
<th>25°</th>
<th>30°</th>
<th>35°</th>
<th>40°</th>
<th>40° 32.5°</th>
<th>45°</th>
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<td>0.05</td>
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<td>1.83</td>
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<td>2.83</td>
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<td>2.49</td>
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The differential equation (6th) is entirely satisfied for the case, which is specified with the Gothic type.
Buckling and Failure of Thin Rectangular Plates in Compression.

We shall next observe more closely the distribution of compressive stress. We find that the compressive stress at the central part of any cross section parallel to the edges subjected to compressive load suddenly diminishes when the critical point of buckling is passed and becomes negative (i.e. tensile stress) when $\frac{e}{e_0}$ is greater than 4.65 (Fig. 5). This absurdity appears not only in the approximate solutions, but also in the exact solution at $\frac{e}{e_0} = 11.57$. We may, therefore say with certainty that this awkwardness in no way implies inaccuracy of the approximate solution (8). We should find another reason to explain it.

The most important assumption taken as the basis of the present theoretical analysis are the partial differential equation (6) and Cox's condition of uniform overall strain (7), which will, when closely discussed, present the question as follows:

In the first article we have neglected three of the six general equations of equilibrium (1), (2) under the bold presumption that the deflection $w$ of the middle surface of the plate is so small that the products of its derivatives may be neglected, whereas our problem is
concerned with the state of considerably large deflection. It has been observed that the simple theory of small deflections becomes inaccurate when the deflections are of the order of the magnitude of the thickness. We may easily point out two (formal) weak points of the small deflection theory which will cause such discrepancy,

1. That the wave form is altered from that of the original small deflection wave when the amount of the deflections becomes considerable;

* An interesting analogy will be noticed here between Fig. 4 and the stress-strain diagram of the tensile test of the material, say mild steel. Since the abscissa and ordinate of Fig. 4 express the relative overall strain and mean compressive stress respectively, Fig. 4 may be regarded as a stress-strain diagram theoretically obtained for a test piece in plate form. It may be regarded as an example of the general tendency in stress-strain diagrams for all similar buckling problems. Now the critical point of buckling, \( e = e_0 \), appears to correspond to the yield-point of the material, and we can observe a similarity in the stress-strain diagram of mild steel to the theoretical stability curve in Fig. 4, both above and below the critical point. We may therefore suppose that the yielding of material may be a problem of stability analogous to any kind of buckling of elastic bodies, as already pointed out by K. Nakanishi. (Rep. of the Aer. Res. Inst. T. I. U. Vol. VI. No. 72 P. 83 1931). Our present result consequently appears to afford datum in confirmation of Nakanishi's theory regarding the yielding of material.
2. That the influence of the terms of the second order which has been neglected as small, becomes so great as to cause an inaccuracy of the theory.

The former of these effects has been taken into consideration in our theory. The main difficulty which we are met with is therefore to know the manner and amount of the latter effect.

Now making use of (12) we will compare the order of the amount of the quantities in the first members of the original equations of equilibrium of the plate element (1) and (2). The amount of the quantities of the first members of the first two equations of (1), which have been neglected as small in our theory, divided by that of the third are found to be of the order of $A\sqrt{\varepsilon_0}$, which varies with $\sqrt{\varepsilon_0} = \frac{\pi}{\sqrt{3}(1-\sigma^2)} \frac{h}{d}$ and becomes very small when the thickness of the plate is very small compared with the breadth, $A$ being a constant as calculated above. Hence we may calmly neglect these equations without any uncertainty when the thickness to breadth ratio of the plate is sufficiently small. The third equation of (2) is neglected upon a similar consideration. Thus the difficulty of the second assumption has lost its generality.

From these considerations we may conclude that the large deflection effect, which makes the simple theory of small deflection unreliable when the amount of the deflections are of the order of the magnitude
of the plate thickness, is chiefly attributed to the varying of the wave form connected with the condition (7), the effect of the neglected differential equations being very small, as far as the original establishment \( T_1 = \frac{\partial T_2}{\partial y} = S_1 = -S_2 = 0 \) is permitted, in the case when the plate is very thin and the amount of its deflections are of the order much higher than the plate thickness.

When the thickness to breadth ratio is considerably large, the second effect must also be taken into consideration. Fortunately, however, we have some reason, as explained later, to believe, that the amount of errors, introduced by it will not be so unfavorably large as to disturb the usefulness of our theory.

At the same time, however, since we ourselves have also assumed the wave form in \( y \)-direction to be a sine curve, this may impose certain slight restriction of the solution of our problem, thus, inevitably introducing certain slight errors. Strictly speaking, the wave form in \( y \)-direction should also be expressed in a more suitable form such as Fourier's series, of which the sine wave would be a considerably fair approximation. But, as we have shown in the last article, the differential equation (6) is entirely satisfied at \( \frac{e}{e_0} = 0 \) and \( \frac{e}{e_0} = 11.51 \) by the sine wave. We may, therefore say with some certainty that the deviation from the sine wave is not remarkable in the range of our numerical calculation.

After all above considerations, returning to the question of the awkwardness of the stress distribution, we can conclude that this awkwardness rather suggests inadequacy of Cox's original assumption that the plate is subjected to uniform compressive strain. If we were permitted to indulge in bold presumption about actual conditions, it would appear that the central parts of the compressed edges do not touch the straining rigid body any more beyond a certain critical state. That is to say, the edges of the rigid body and the plate are no longer in contact throughout their lengths, the middle part of the plate edge having slightly receded (shrank) forming a gap. From this point of
view it would appear meaningless to apply the condition of uniform overall strain for large values of \( \frac{e}{e_0} \) in the case of a rectangular plate simply supported at its four edges. The range of numerical calculation is therefore confined to \( \frac{e}{e_0} \leq 15.84 \), outside of which the cross section of the buckled plate would make things awkward, becoming concave upwards at the central parts while it is convex at the parts nearer to the edges (Fig. 6), an absurdity that lends further weight to our criticisms just made.

\[
\frac{\phi}{A} \frac{2K_0}{\pi} \sn \frac{2K_0}{\pi} \frac{2K_0}{\pi} \text{ at } k = \sin \theta = 50^\circ
\]

Fig. 6.

4. Failure of the Plate.

We must next see to what extent the final collapse of the plate is influenced by the bending of it as the result of buckling.

If the components of the fibre stress due to bending in the directions Ox and Oy be denoted by \( f_{bx} \) and \( f_{by} \) respectively, we can then write

\[
f_{bx} = -\frac{Ee_0}{2} \sqrt{\frac{3}{1-\sigma}} \left( \frac{d^2\phi}{d\phi^2} - \sigma \phi \right) \sin \frac{\pi y}{2d} ,
\]
\[ f_{by} = -\frac{E e_0}{2} \sqrt{\frac{3}{1-\sigma^2}} \left( \sigma \frac{d^2 \Phi}{d\theta^2} - \frac{\Phi}{4} \right) \sin \frac{\pi y}{2d}, \]

Hence both \( f_{bx} \) and \( f_{by} \) will be largest at \( \sin \frac{\pi y}{2d} = 1 \), the maximum bending stress occurring at the loop of the wave in \( y \)-direction, while on the other hand the direct compressive stress will be

\[ f_c = -E e_0 \left( \frac{e}{e_0} - \frac{\Phi^2}{4} \right). \]

Therefore, the resultant compressive stress at the loop of the wave in \( y \)-direction is

\[ f_r = -\frac{E e_0}{\sqrt{1-\sigma^2}} \left\{ \left[ \frac{d^2 \Phi}{d\theta^2} - \frac{\Phi}{4} \right]^2 \right\}^{1/2} \]

\[ + \left[ \left( \frac{d^2 \Phi}{d\theta^2} - \frac{\Phi}{4} \right) - 2 \sqrt{1-\sigma^2} \left( \frac{e}{e_0} - \frac{\Phi^2}{4} \right) \right]^{1/2} \]

\[ F = -\frac{f_r}{\sqrt{1-\sigma^2}} \frac{E e_0}{\pi} \]

\[ = \left\{ A^2 s n^3 \frac{2K\theta}{\pi} - \frac{2K\theta}{\pi} \left[ 1 + 4 k^2 - 6 k^2 s n^2 \frac{2K\theta}{\pi} \right] \left( \frac{2K\theta}{\pi} \right)^2 + \sigma \right\}^{1/2} \]

\[ + \left[ -A s n \frac{2K\theta}{\pi} \left( 1 + 4 k^2 - 6 k^2 s n^2 \frac{2K\theta}{\pi} \right) \left( \frac{2K\theta}{\pi} \right)^2 + 1 \]

\[ - 2 \sqrt{1-\sigma^2} \left( \frac{e}{e_0} - \frac{\Phi^2}{4} \right) \right\}^{1/2} \]

The numerical values of \( F \) are calculated and shown in Fig. 7, showing that the maximum resultant stresses exceed by far the stresses at the side edges, where the plate elements are subjected to pure compression alone. It indicates the inadequacy of the idea proposed by Sechler and Cox that failure of the panel subjected to edge compression is anticipated from the compressive stresses at the side edges. The final collapse of
the panel would appear to occur when the maximum resultant skin stress due to combined bending and compression reaches a certain amount, the yield point stress of the material, for example. Comparison with the results of experiments on the failure of duralumin sheets carried out in America by L. Schuman and G. Back(1), however, confirms this rather reckless presumption of ours. As shown in Fig. 8, the experiment agrees very well with the theoretical curve, except near the critical point, where collapses occur at lower loads than anticipated from the theory, which may be attributed to the initial excentricity of the tested plates. Should such initial excentricity exist, the observed crippling loads will be lower than the theoretical ones, with consequence collapsing loads should the failures occur near the critical point.

Fig. 7.


It is desirable to carry out some essential tests on plates of various material in order to obtain more reliable confirmation of the present theoretical analysis. A series of experimental investigations with plates of various light alloys such as are commonly used in aeroplane structures, is now in course of preparation at our Institute.
The permissible edge load for the given dimensions and properties of the material for the plate is easily read from Fig. 8, the use of which in practice is recommended.

![Graph showing the relationship between edge load and normalized stress](image)

**Fig. 8.**

A question remains yet, about the order of the amount of the thickness to breadth ratio of the plate at the collapsing state. As we have shown in the last article we cannot have rely upon the accuracy of our results of calculations, when the plate is too thick. For a duralumin sheet which is predicted from Fig. 8 to collapse at $\frac{t}{e_0} = 15.84$, i.e. the upper end of the interval of $\frac{t}{e_0}$ determined before, we get $AV't_0 = 0.15$ at the predicted collapsing points, which shows us that we must expect of such a collapsing panel an error of the order of 10%. It is interesting to observe that the order of this error does not vary remarkably at one of the two half intervals of $\frac{t}{e_0}$ farther to the critical.
point of buckling than the other. Therefore our theoretical calculation does not hold very well at the collapsing state although it is all right when the straining of the plates is far before the collapsing points. But the error of 10% is not so grave in practice of strength calculation as to destroy the usefulness of Fig. 8.

5. Summary.

(a) If a rectangular plate were subjected to compression at two of its supported edges, the final solution of the problem will be to solve the ordinary differential equation (6''), although certain restrictions are not entirely absent.

(b) The solution of (6'') for a rectangular plate simply supported at its four edges is given by (8). The wave form and the distribution of the edge stress anticipated from the present solution (8) show the same tendency as in Cox's approximate assumption. But they are much freer from unsatisfactory assumptions.

(c) The condition of uniform overall strain is not always satisfied for the merely supported plate. The edges of the straining rigid body and the plate are no longer in contact throughout their lengths; the middle part of the plate edge having slightly receded (shrunk) forming a gap, when the overall strain of the plate exceeds $4.65 \varepsilon_0$. It therefore seems purposeless to apply the equation (8) for too large a value of $\frac{\varepsilon}{\varepsilon_0}$.

(d) In practical problems, however, the parts that are nearer to the side edges play the most important rôle, whether in determining the plate load or the collapse point of the plate. Therefore, for such practical purposes, we may use equation (8) with a fair degree of certainty.

(e) The final collapse of the plate occurs when the combined stress of bending and direct compression attains a certain amount. And since the maximum fibre stress due to bending is as great as the direct com-
pressive stress, the influence of curving of the buckled plate is very marked, in consequence of which the wave form after buckling should be expressed as accurately as possible. From this point of view, the authors' solution here may be regarded as an advance on the old method, since it is deduced from the general equations of equilibrium of plate elements, without previously assuming the wave form.

(f) For practical purposes it is advisable to use Fig. 8, from which can be read at once the permissible load for any given rectangular plate simply supported at its four edges.

The present method is believed to be more reliable than the other methods that have been proposed by various authors so far.

October 22, 1934.

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