Base Pressure on Two-Dimensional Blunt-Trailing Edge Wings at Supersonic Velocities

By

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Summary. Present paper gives the results of an analytical approach to the base pressure on two-dimensional blunt-trailing edge wings at supersonic velocities, when the approaching boundary layer is laminar. Interaction between dissipative flow and neighbouring nearly isentropic flow behind the flow separation at the blunt-trailing edge is first analysed and the quantitative relations concerning the viscous interaction mechanism are presented. In contrast to the exact inviscid flow theory which gives an infinite number of possible solutions of the base pressure, it is shown that the viscous interaction mechanism permits only one solution available for the given boundary conditions. Finally, the numerical computation is carried out for free stream Mach numbers of 1.5, 2.0 and 3.1 to obtain the base pressure on two-dimensional thin airfoil with blunt-trailing edge and the results are compared with the experiment made by Chapman, Wimbrow and Kester. The agreement between the present theory and experiment is fairly good.

SYMBOLS

\( x, y \) coordinates system
\( x^*, y^* \) non-dimensional coordinates system normalized by \( \delta e \)
\( \delta e \) characteristic length corresponding to the thickness of the jet at the trailing edge
\( l \) distance from trailing edge to the shock measured along \( x \)
\( h \) height of the base
\( c \) chord length of the airfoil
\( t \) thickness of the airfoil
\( u, v \) components of velocity vector in the viscous region
\( U \) velocity of the inviscid flow
\( U_{\text{max}} \) maximum velocity of the inviscid flow
\( \rho \) density
\( p \) pressure
\( T \) temperature
\( M \) Mach number
\( Re \) Reynolds number
\( \mu \) coefficient of viscosity
\( \nu \) kinematic viscosity
\( \varepsilon \) eddy kinematic viscosity

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\[ C_{pb} \quad \text{base pressure coefficient defined as} \quad C_{pb} = \frac{p_b - p_{in}}{\frac{1}{2} \rho_{in} U_{in}^2} \]

\[ \frac{c}{h \sqrt{Re_c}} \quad \text{Reynolds number shape parameter} \]

**Subscripts:**
- \( a \) condition in approaching flow
- \( \infty \) condition in free stream
- \( 2 \) condition in the inviscid flow outside the jet mixing region
- \( 0 \) condition referred to the stagnation

**INTRODUCTION**

The base pressure problems have been investigated by both analytical and experimental methods over a long period of time so that a noticeable amount of information is available in the papers \([1]\) \([2]\) pertinent to external and internal-flow configurations. However, it is well known that the exact inviscid-flow theory concerning the base pressure leads to an infinite number of possible solutions satisfying all boundary conditions at any free stream Mach number. Chapman \([4]\) pointed out that there existed a limiting solution, which was determined by the maximum deflection angle of the flow through single oblique shock wave. Moreover, he showed that, for the particular case of a body of revolution having no sting attached, only one solution was possible in an inviscid flow, but it corresponded to zero base drag. Accordingly, he concluded that a exact inviscid-flow theory could not be satisfactory for the practical application.

Since the inviscid-flow theory does not adequately describe the conditions of a real fluid flow, it has long been recognized that mixing in the wake just aft of the body must determine the amount of recompression that can be supported by the flow and, hence, the base pressure, but this concept has never been precisely formulated. From the experimental point of view, Chapman \([7]\) reviewed the entire problem and found that existing data on the base pressure could be correlated in terms of the ratio of the boundary layer thickness at the base to the base diameter without explaining the mechanism of correlation.

More recently, Crocco and Lees \([3]\) gave an analytical solution of the base
pressure derived from the discussion on relations of the boundary layer properties near separation point, in which the interaction between dissipative flow and neighbouring nearly isentropic stream was taken to be an essential physical process. By their analysis the correlation was reasonably established between the base pressure and the ratio of boundary layer thickness at the base to the base diameter, and the qualitative agreement was found between these theoretical predictions and calculations and the base pressure data of Bogdonoff [4] and Chapman [7] on blunt-based bodies of revolution and Chapman's data [5] on supersonic airfoils with blunt-trailing edge. It must, however, be noted that the theoretical approach proposed by Crocco and Lees is semi-empirical in the sense that it includes an empirical constant of mixing coefficient. This arises from the fact that the turbulent flow theory gives no rigorous information to the eddy kinematic viscosity. In this circumstance Crocco and Lees adopted 0.03 for the mixing coefficient in their numerical calculation of the base pressure. This value seems to be reasonably supported by the experimental results proposed by Schubauer and Klebanoff [6]. Nevertheless, the theoretical value of base pressure thus calculated is far beyond the experimental ones. However, this quantitative disagreement between theory and experiment does not seem to arise from the incorrect evaluation of mixing coefficient, but it seems to suggest that the detailed flow structure in the jet mixing region contributes to the determination of the base pressure to a certain extent.

Since the exact inviscid-flow theory does not give a unique solution of the base pressure for the given boundary conditions, Korst [7] proposed an additional criterion for reattachment of the separated flow which gives a compatibility between jet mixing region and the amount of recompression downstream of the base. This criterion can be interpreted into the condition for conservation of mass of fluid in the dead-air region behind the base. By the use of this criterion Korst gave a theoretical solution of base pressure on the two-dimensional supersonic step without using any empirical values. However, this theoretical completeness seems to be naturally resulted, because it is valid only in the limiting case when Reynolds number tends to infinity and, consequently, the mixing coefficient is zero. Since, as pointed out by Chapman, the base pressure is highly dependent on Reynolds number for the laminar approaching boundary layer, it is of great value to clarify the mechanism and correlation between dissipative region and neighbouring nearly isentropic flow.

It is the purpose of the present paper to give a theoretical approach to the interaction between the dissipative flow and the nearly isentropic flow in mathematically simpler form, which may be further applied to the problem of base pressure on two-dimensional blunt-trailing edge wings at supersonic velocities, when the approaching boundary layer is laminar. For the purpose of simplifying the analysis a simplified model of the flow pattern is first constructed and the interaction in the jet mixing region is analysed by the use of Prandtl's mixing length hypothesis together with some other assumptions. Although this mechanism involves primarily the eddy kinematic viscosity which still remains to be
an empirical constant in the present approach, some quantitative relations concerning the interaction phenomenon are derived. These relations are then applied to the base pressure problem with the aid of Korst’s criterion. For the purpose of confirming the present theory numerical computations are made for free stream Mach numbers of 1.5, 2.0 and 3.1 and the results are compared with experimental data obtained by Chapman, Wimbrow and Kester. It is shown that the quantitative agreement is good between the present theory and experiment.

1. **Simplified Model of Flow Pattern and Basic Assumptions**

When two-dimensional airfoils with blunt-trailing edges are in the uniform flows at supersonic velocities, it is well known that there appears a Prandtl-Meyer fan at the corner of the base and this decreases the pressure on the base which results in base drag. From the experimental point of view, the flow separation at the corner of the base seems to induce the transition of viscous layer to turbulence even for the laminar approaching boundary layer on the airfoil surface and this turbulent mixing region spreads itself downstream like a jet, while the nearly isentropic flow outside the dissipative region is almost uniform. Since the flow must become parallel to the undisturbed flow far downstream of the airfoil, overexpanded flow by Prandtl-Meyer fan must be recompressed by one or more shocks. Consider the simplest case in which the airfoil surface is parallel to the flight direction at the trailing edge, then the characteristic feature of such a flow pattern is such as shown in Fig. 1. In the jet mixing region behind

![Diagram](image)

**Figure 1.** Simplified model of flow field.

the base there are three characteristic boundaries of the flow such as jet boundary, separating stream line and dividing stream line. The jet boundary separates the region of dissipative flow from the neighbouring nearly isentropic flow. The separating stream line is defined as a stream line that separates the fluid included initially in the viscous layer at the trailing edge from the fluid that enters into the dissipative region from the external isentropic flow by jet mixing. The dividing stream line separates the fluid included initially in the approaching flow from that in the dead-air region.
Strictly speaking, it may be reasonable to consider that there exists a couple of vortices in the dead-air region. The solution in the jet mixing region must be, therefore, continued to the solution of the vortex inside the dividing stream line in such a way that the velocity and shear stress are continuous across the dividing stream line. However, as has been already analysed by Crocco and Lees, the base pressure depends mainly upon the mass flow entering into the jet mixing region through approaching boundary layer. Since this mass flow has a larger dynamic pressure than that of the low speed vortex in the dead-air region, it will be intuitively recognized that the slight change in condition of the dead-air region has less contribution to the entire flow field. This characteristic will make it reasonably possibly to assume that the fluid in the dead-air region is at rest as a first-order approach to the problem under consideration. And this simplification seems to be supported by the fact that the theoretical results proposed by Crocco and Lees have a qualitative agreement with the experimental ones obtained by Chapman and others.

On the basis of the simplified model of the flow field the subsequent analysis will be made together with some basic assumptions which may be summarized as follows:

1. the approaching boundary layer is laminar and the transition point is situated at the trailing edge,
2. pressure gradient in the jet mixing region is negligible,
3. viscous region is isoenergetic and Prandtl number is unity,
4. mixing coefficient is constant everywhere in the mixing region,
5. there is no heat transfer in the dead-air region,
6. the fluid in the dead-air region is at rest,
7. the airfoil is so thin that the approaching conditions of the inviscid flow can be replaced by the free stream conditions.

Strictly speaking, the mixing coefficient must be a function of \( \phi \) in which overall Reynolds number and Mach number of the external inviscid flow are included as parameters. For the case of laminar mixing, the mixing coefficient is explicitly formulated by Crocco and Lees [3], while it is not yet done precisely for the case of turbulent mixing because of difficulty in theoretical treatment of the turbulent flow. The experiment, however, reveals that the turbulent mixing coefficient is of order of ten times the laminar value. This fact seems to be largely responsible for the marked difference between the turbulent and laminar interaction phenomena. The constant mixing coefficient assumed in the present approach should be regarded as an average value.

2. TURBULENT JET MIXING OF A COMPRESSIBLE FLUID

Since the base pressure problem is complicatedly connected with the viscous interaction accompanied with the flow separation at the corner of the base, it is first required to clarify the details of flow structure in the dissipative region. In such an interaction between the dissipative flow and the nearly isentropic stream,
the 'external flow' cannot be regarded as a known datum for the calculation of 'internal' dissipative flow. In contrast to usual Prandtl boundary layer theory, the development of the dissipative flow itself helps to determine the external flow, and this interplay makes the inviscid flow theory unable to determine the base pressure uniquely for the given boundary conditions. It is the aim of this section to bring out the importance of the transport of momentum from the outer stream to the dissipative flow (turbulent mixing) in determining the flow pattern in the jet mixing region and to formulate this concept in quantitative terms on the basis of the simplified model of flow field and assumptions mentioned in last section.

Let the origin of coordinates system $(x, y)$ be taken at the point at which $u = U_y/2$ at the trailing edge, $x$-axis being aligned with the line on which $u = U_y/2$, $y$-axis being normal to $x$-axis, where $u$ and $U_y$ denote $x$-component of the local velocity vector and the velocity of the external inviscid flow, respectively. In this coordinates system the basic equations applied to the jet mixing region are given as

\begin{equation}
\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{2.1}
\end{equation}

\begin{equation}
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y}, \tag{2.2}
\end{equation}

where $\tau$ denotes turbulent shear stress. It must, however, be noted that all physical quantities in the above equations and in the equations that will appear in the subsequent analysis indicate their time-mean values. By the use of Prandtl's mixing length hypothesis the turbulent shear stress can be expressed as

\begin{equation}
\tau = \varepsilon \rho \frac{\partial u}{\partial y}, \tag{2.3}
\end{equation}

where $\varepsilon$ denotes eddy kinematic viscosity which is given by

\begin{equation}
\varepsilon = \kappa c_1 U_y, \tag{2.4}
\end{equation}

where $\kappa$ and $c_1$ indicate the constant of proportionality and the width of the mixing zone, respectively. Since, in usual turbulent boundary layer theory, $b$ is assumed to be proportional to $x$, the eddy kinematic viscosity can be expressed as

\begin{equation}
\varepsilon = \kappa c_1 U_y x, \tag{2.5}
\end{equation}

where $c_1$ is another constant of proportionality. By the use of Eqs. (2.3) and (2.5) the momentum equation can be written as

\begin{equation}
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \varepsilon \frac{x}{L} \frac{\partial}{\partial y} \left( \rho \frac{\partial u}{\partial y} \right), \tag{2.6}
\end{equation}

where $L$ denotes a characteristic length in the jet mixing region and

\begin{equation}
\varepsilon = \kappa c_1 U_y L. \tag{2.7}
\end{equation}

The boundary condition are given as

\begin{equation}
u(x, 0) = \frac{1}{2} U_y, \quad u(x, \infty) = U_y, \quad u(x, -\infty) = 0. \tag{2.8}
\end{equation}
Introducing the non-dimensional expressions and stream function, \( \psi \), such as

\[
\begin{align*}
    u^* &= \frac{u}{U_2}, \quad v^* = \frac{v}{U_2}, \quad \rho^* &= \frac{\rho}{\rho_2}, \\
    x^* &= \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad T^* = \frac{T}{T_2}, \\
    \psi^* &= \frac{\psi}{\sqrt{\mu_2 \rho_2 U_2 L}}, \quad \epsilon_2^* = \frac{\epsilon_2}{\nu_2},
\end{align*}
\]

then, the momentum equation can be reexpressed as

\[
\frac{\partial u^*}{\partial x^*} = \epsilon_2^* x^* \frac{\partial}{\partial \psi^*} \left( \rho^* u^* \frac{\partial u^*}{\partial \psi^*} \right), \tag{2.9}
\]

where

\[
\frac{\partial \psi^*}{\partial x^*} = -\sqrt{\frac{U_2 L}{\nu_2}} \rho^* u^*, \quad \frac{\partial \psi^*}{\partial y^*} = \sqrt{\frac{U_2 L}{\nu_2}} \rho^* u^*. \tag{2.10}
\]

The boundary conditions are also transformed into

\[
u^*(x^*, 0) = \frac{1}{2}, \quad u^*(x^*, \infty) = 1, \quad u^*(x^*, -\infty) = 0. \tag{2.11}
\]

Existence of similar solution for Eqs. (2.9) and (2.11) has already been proved by Görtler [8] theoretically and by Reichardt [9] experimentally in the case of incompressible jet mixing and also by Goederum, Wood and Brevoort [10] experimentally in the case of supersonic jet mixing of a compressible fluid. Therefore, the existence of a similar solution is assumed in the present approach. Putting the stream function to be

\[
\psi^* = \sqrt{\epsilon_2^*} x^* f(\xi), \tag{2.12}
\]

where

\[
u^* = \frac{1}{2} f'(\xi), \tag{2.13}
\]

analogously to the case of incompressible jet mixing, then the momentum equation can be expressed as

\[
\frac{d}{d\xi} (\rho^* f'') + 2 f f'' = 0, \tag{2.14}
\]

where

\[
f'(0) = 1, \quad f'(\infty) = 2, \quad f'(-\infty) = 0. \tag{2.15}
\]

In order to solve Eq. (2.14) under the boundary conditions, Eq. (2.15), \( \rho^* \) in Eq. (2.14) must be expressed in terms of \( f'(\xi) \). In the analytical treatment of the turbulent boundary layer in compressible fluid, Van Driest [11] gave a proof that the same energy equation as in the case of compressible boundary layer on an insulated flat plate is valid under the assumptions (2), (3) and (5) mentioned in Section 1. Hence,

\[
T^* = 1 + \frac{r - 1}{2} M_2^2 (1 - u^*). \tag{2.16}
\]

Introducing the Mach number parameter, \( \epsilon_2 \), defined by
\[ \varepsilon_z = \frac{U_0}{U_{\text{max}}} = \left[ 1 + \frac{2}{(r-1)M_0^2} \right]^{-\frac{1}{2}}. \]  

(2.17)

Then, \( \rho^* \) can be expressed as

\[ \rho^* = \frac{1}{T^*} = \frac{1 - \varepsilon_z^2}{1 - \frac{1}{4} \varepsilon_z^2 f'^2}. \]  

(2.18)

Hence, Eq. (2.14) becomes

\[ \frac{d}{d\xi} \left[ \left( \frac{1 - \varepsilon_z^2}{1 - \frac{1}{4} \varepsilon_z^2 f'^2} \right)^2 f''' \right] + 2f f'' = 0, \]  

(2.19)

which is solvable.

The solution of Eq. (2.19) can be obtained by numerical integration, since it is very difficult to get an analytical solution of it. On the other hand, in experiment, Gooderum and others [10] obtained the result that the velocity distribution in the supersonic jet mixing region is almost similar to that of the incompressible jet. This experimental fact seems to suggest the validity of the assumption of \( \rho^* = 1 \) as a first-order approximation for the velocity distribution. In such a case, Eq. (2.19) can be reduced to a simpler form such as;

\[ f''' + 2f f'' = 0. \]  

(2.20)

This equation is the same as the one solved by Görtler [8]. He tried to solve this equation by assuming a series expansion of \( f(\xi) \) such as

\[ f(\xi) = \sum_{n=0}^{\infty} \lambda^n F_n(\xi), \]  

(2.21)

where

\[ F_0(\xi) = \xi, \quad \lambda \leq 1, \]

and showed that the convergency of this series was quite good even for the case of \( \lambda = 1 \). Hence, in the present approach, it seems enough to approximate as

\[ f' = 1 - \text{erf} \xi, \]  

(2.22)

\[ f = \xi + \int_0^\xi \text{erf} \alpha \, d\alpha - 0.1608, \]  

(2.23)

\[ u^*(\xi) = \frac{1}{2} (1 + \text{erf} \xi). \]  

(2.24)

Although these approximate solutions are valid in the case of incompressible jet mixing, it can be possibly expected that they may reasonably describe the flow phenomenon in high speed part of the compressible jet mixing region where \( u^* \) is approximately equal to unity. This consideration leads to an expectation that the base pressure problem might be described by these approximate solutions, since the mass of fluid entered into the jet mixing region from approaching boundary layer will occupy almost high speed part of the dissipative region.
3. SOME QUANTITATIVE RELATIONS ASSOCIATED WITH THE INTERACTION

Since the flow velocity in the dissipative region approaches asymptotically that of the outer nearly isentropic flow as \( y \) increases, there is no distinct jet boundary between them. In this circumstance, the jet boundary is defined in the present approach by the following equation:

\[
 u^*(x^*, y_0^*) = 0.9975
\]  

(3.1)

where \( y_0^* \) denotes the height of the jet boundary. \( u^* \) is a function of only \( \xi \) for the given Mach number and so

\[
 u^*(x^*, y_0^*) = (1/2)f'(\xi_0) = 0.9975.
\]

Hence, by the use of Eq. (2.24) the position parameter for the jet boundary is obtained as

\[
 \xi_0 = 2.150.
\]  

(3.2)

Separating stream line can be defined by the following equation:

\[
 \int_{y_s}^{y_t} \rho u dy = m,
\]  

(3.3)

where \( m \) denotes the mass of fluid entered into the dissipative region from the external flow and \( y_s \) is the height of the separating stream line. If the mass of fluid at the beginning of the jet mixing region is given by \( \rho_2 U_2 \delta \epsilon \) and the characteristic length, \( L \), is chosen to be \( \delta \epsilon \), which denotes effective thickness of the jet at the trailing edge, then, Eq. (3.3) becomes

\[
 \int_{y_s}^{y_t} \rho u u^* dy^* = m = \frac{m}{\rho_2 U_2 \delta \epsilon}.
\]

By the use of the mixing coefficient, \( k(x) \), defined by Crocco and Lees as

\[
 \frac{dm}{dx} = k(x; Re, M_2) \rho_2 U_2,
\]  

(3.4)

\[
 m(0; Re, M_2) = 0,
\]

the above equation can be rewritten as

\[
 \omega x^* \int_{\xi_s}^{\xi_t} f'(\xi) d\xi = \frac{1}{\delta \epsilon} \int_{0}^{x} k(x) dx,
\]

where

\[
 \omega = \sqrt{\frac{\delta \epsilon}{U_2 \delta \epsilon}} = \sqrt{\kappa_t c_1}
\]

From the assumption of constant mixing coefficient the above integral can be re-expressed as

\[
 \omega \{ f(\xi_0) - f(\xi_t) \} = k.
\]  

(3.5)

The separating stream line can be determined by conservation of momentum in the mixing region. Since the momentum at the beginning of the jet must be conserved by the fluid existing inside the separating stream line,
\[ \int_{-\infty}^{\infty} \rho u^2 dy = \text{const}, \]

or
\[ \omega x^* \int_{-\infty}^{\infty} u^* f'(\xi) d\xi = \text{const}. \]  

(3.6)

Differentiation of Eq. (3.6) with respect to \( x^* \) leads to
\[ \int_{-\infty}^{\infty} u^* f'(\xi) d\xi = u^*(\xi) f(\xi). \]  

(3.7)

Numerical calculation of Eq. (3.7) by the use of Eqs. (2.22) (2.23) and (2.24) gives the position parameter for the separating stream line as
\[ \xi_s = 1.339. \]  

(3.8)

It is a remarkable result that the position parameter for the separating stream line is independent of \( x \) and the flow velocity along the separating stream line is constant. This arises from the assumption of constant mixing coefficient.

The dividing stream line can be obtained in such a way that the mass of fluid existing between dividing stream line and the separating stream line must be equal to that at the beginning of the jet. Hence,
\[ \int_{x_s}^{y_i} \rho u dy = \rho_a U_a \delta e, \]

or
\[ \omega x^* \{ f(\xi_s) - f(\xi_a) \} = 1. \]  

(3.9)

Elimination of \( \omega \) from Eqs. (3.5) and (3.9) gives
\[ k x^* \frac{f(\xi_s) - f(\xi_a)}{f(\xi_a) - f(\xi_s)} = \delta e. \]  

(3.10)

Since \( f(\xi_s) \) and \( f(\xi_a) \) in Eq. (3.10) are independent of \( x \), \( f(\xi_a) \) must be dependent on \( x \). This expresses a noticeable fact that the velocity, \( u^* \), is variable along the dividing stream line.

The characteristic length, \( \delta e \), which corresponds to the effective thickness of the jet at the trailing edge can be related to the thickness of the approaching boundary layer at the base. From the conservation of mass,
\[ \rho_a U_a \delta e = \rho_a U_a (\delta_a - \delta_a^*) \]

where \( \delta_a \) and \( \delta_a^* \) denote thickness and displacement thickness of the approaching boundary layer at the base, respectively, and subscript \( a \) indicates approaching condition. From the usual boundary layer theory, \( \delta_a \) and \( \delta_a^* \) are given approximately as
\[ \delta_a = 5.0 \sqrt{\frac{\nu_a}{U_a}}, \quad \delta_a^* = 1.73 \sqrt{\frac{\nu_a}{U_a}}. \]

Hence, \( \delta e \) can be expressed as
\[ \delta e = 3.27 \frac{c}{\sqrt{Re_a}} \frac{M_a}{M_a} \left( \frac{5 + M_a^2}{5 + M_a^2} \right)^{\frac{3}{2}}. \]  

(3.11)
where $Re_c$ denotes Reynolds number referred to the chord length, $c$, of the airfoil and approaching stream condition.

If the width of jet mixing region is assumed to be small compared with the height of the base, $h$, then, the distance, $l$, between trailing edge and the entrance of recompression region (See Fig. 1) measured along $x$ can be expressed approximately as

$$
l \approx \frac{h}{2 \sin \theta},
$$

(3.12)

where $\theta$ denotes turning angle of the inviscid flow by Prandtl-Meyer fan at the corner of the base. Hence, by the use of Eqs. (3.11) and (3.12), Eq. (3.10) can be expressed at the entrance of the recompression region as

$$
\frac{c}{h} \frac{1}{\sqrt{Re_c}} = 0.1482 \frac{M_a}{Ma} \left( \frac{5 + M_a^2}{5 + M_\infty^2} \right)^{\frac{k}{2}} \frac{k}{\sin \theta} f(\xi) - f(\xi_{a,1}),
$$

(3.13)

where $f(\xi_{a,1})$ is the value of $f(\xi_a)$ at the entrance of the recompression region. Eq. (3.13) indicates the interrelation between the condition in the dissipative region denoted by $f(\xi_{a,1})$ and the condition in the neighbouring nearly isentropic region subscribed by 2. Furthermore, as is seen in the equation, this interplay is intimately connected with the approaching condition subscribed by $a$ and Reynolds number and also with the shape of the body boundary. Eq. (3.13) clearly indicates that the effects of Reynolds number and shape of the body boundary contribute to the condition in nearly isentropic flow aft of the Prandtl-Meyer expansion in a combined form of $\frac{c}{h} \frac{1}{\sqrt{Re_c}}$, when the approaching boundary layer is laminar. This result together with that of Crocco and Lees [3] confirms the correlation found out by Chapman [7] experimentally.

4. **Base Pressure on Blunt-Trailing Edge Wings**

In general, it is believed that the base pressure must be expressed in terms of various parameters such as approaching Mach number, Reynolds number, Prandtl number, thermal conditions, shape of the body boundary, angle of attack, etc. However, if the assumptions such as made in Section 1 are introduced in the treatment of the problem, some of the parameters are reduced to

$$
p_b = p\left(M_a, Re_c, \frac{t}{c}, h, \alpha \right),
$$

(4.1)

where $t/c$ and $\alpha$ denote thickness-chord ratio of the wing and boat-tail angle, respectively, and $h$ indicates the height of the base. Moreover, from the experimental point of view, the base pressure seems to be less influenced by thickness-chord ratio and boat-tail angle. This experimental fact indicates that, if the airfoil surface is assumed to be parallel to the undisturbed flow at the trailing edge, then, the base pressure depends mainly upon approaching Mach number, Reynolds number and the height of the base as

$$
p_b = p(M_a, Re_c, h),
$$
and, by the use of Chapman's correlation, the above equation may be further reduced to

\[ p_b = p\left( M_a, \frac{c}{h \sqrt{Re}} \right). \]  \hspace{1cm} (4.2)

Eq. (4.2) is a formal expression of the base pressure corresponding to the case of zero boat-tail angle such as shown in Fig. 1. The effect of boat-tail angle will be discussed later.

As has been already mentioned previously, the exact inviscid flow theory gives an infinite number of possible solutions to the base pressure. The inviscid flow theory requires only one boundary condition that the flow must be parallel to the undisturbed flow after the recompression, and, consequently, the base pressure is expressed as a function of approaching Mach number only. The fact that any more restriction is not imposed on the flow field enables to determine an infinite number of possible solutions satisfying the boundary condition mentioned above for a given approaching Mach number, as pointed out by Chapman.

In contrast to the inviscid flow theory which is inadequate to predict the real phenomenon of base pressure, the viscous interaction mechanism gives a definite interrelation between dissipative flow and neighbouring nearly isentropic stream at the entrance of the recompression region such as formulated in Eq. (3.13), which shows that the Mach number of the external inviscid flow is a function of approaching Mach number, Reynolds number and shape of the body boundary. In this sense, in the order of accuracy of the present approximation, Eq. (4.2) should be consistent with Eq. (3.13). Moreover, it is remarkable that the jet mixing region in viscous interaction mechanism requires an additional condition to be imposed on the wake behind the recompression. This condition does not permit any arbitrary pressure behind the shock and, therefore, it becomes possible to determine the base pressure uniquely for the given conditions of the approaching flow field.

The additional condition just mentioned above is the conservation of mass of fluid in the dead-air region. As a condition satisfying the above requirement Korst [7] proposed a criterion for reattachment of the separated turbulent flow such that, at the entrance of the recompression region, the fluid along the dividing stream line must have a level of mechanical energy expressed by the stagnation pressure, \( p_{0a,1} \), equal to the static pressure behind the shock, \( p_s \), so that recompression to the static pressure \( p_s \) is possible by the complete conversion of the kinetic energy. Physical formulation of this criterion is given by

\[ \frac{p_{0a,1}}{p_a} = \frac{p_s}{p_a}. \]  \hspace{1cm} (4.3)

Interpretation of Eq. (4.3) leads to the statement that the fluid having \( p_0/p_s < 1 \) at the entrance of the recompression will not be able to penetrate into the region where the static pressure \( p_s \) prevails, while the fluid having \( p_0/p_s > 1 \) will pass through the recompression region. This criterion seems to be reasonable and is a useful one, because it relates a part of dissipative region to Rankine-Hugoniot
relation in the external inviscid flow.

The left-hand side of Eq. (4.3) is expressible by using the reversible adiabatic relation as

$$\frac{p_{a\text{-e}}}{p_{a}} = \left(1 + \frac{\gamma - 1}{2} M_{a\text{-e}}^2\right)^{\frac{\gamma}{\gamma - 1}},$$  \hspace{1cm} (4.4)$$

where $M_{a\text{-e}}$ indicates Mach number of the dissipative flow at the entrance of the recompression region on the dividing stream line, which can be obtained by the following equation:

$$M_{a\text{-e}}^2 = \frac{u_{a\text{-e}}^2}{a_{a\text{-e}}^2}$$
$$= M_e^2 \frac{u_{a\text{-e}}^2}{a_{a\text{-e}}^2} \rho_{a\text{-e}}$$
$$\frac{1}{1 - \varepsilon_2^2} M_e^2 \frac{u_{a\text{-e}}^2}{1 - \varepsilon_2^2}. \hspace{1cm} (4.5)$$

On the other hand, the static pressure ratio across the shock is given by Rankine-Hugoniot relation as

$$\frac{p_s}{p_{a}} = 1 + \frac{2\gamma}{\gamma + 1} (M_s^2 \sin^2 \beta - 1), \hspace{1cm} (4.6)$$

where $\beta$ denotes shock angle, which is given by

$$\tan \theta = \cot \beta \frac{2(M_s^2 \sin^2 \beta - 1)}{M_e^2(\gamma + \cos 2\beta) + 2}. \hspace{1cm} (4.7)$$

Since the flow deflection angle, $\theta$, across the shock must be equal to the turning angle of the inviscid flow through Prandtl-Meyer fan, the shock angle $\beta$ can be determined as a function of $M_s$. This, in turn, determines $u_{a\text{-e}}^2$ as a function of $M_s$ by use of Eq. (4.3) and, consequently, $f(\xi_{a\text{-e}})$ in Eq. (3.13). Hence, the right-hand side of Eq. (3.13) can be expressed as a function $M_s$ and $M_e$.

As has been just summarized above, Korst's criterion connects the recompression through shock uniquely with the viscous mixing region. Therefore, the Mach number of the nearly isentropic flow outside the dissipative region, which corresponds to the base pressure, can be determined uniquely for the given approaching conditions such as $M_s$ and \[ c \frac{1}{h \sqrt{Re_c}}. \]

![Figure 2. Simplified flow model with boal-tail angle.](image-url)
It remains to discuss the effect of boat-tail angle on the base pressure. The experimental results obtained by Chapman, Wimbrow and Kester [3] indicate that, if the boat-angle is small, it has little contribution to the base pressure for a fixed free stream Mach number. This fact can be briefly recognized under the following consideration: consider an airfoil model with an after-body at boat-tail angle, \( \alpha \), aft of the parallel surface to the undisturbed flow as shown in Fig. 2. In this case it will be easily found that the approaching Mach number must be larger than that in the case of zero boat-tail angle, because the flow must be pre-expanded by the Prandtl-Meyer fan at point \( A \) in the figure. Since Eq. (3.13) is valid irrespective of the boat-tail angle, it is expressible in this case as

\[
\left( \frac{c}{h} \right)_{\text{Re}^{\alpha=\alpha}} = 0.1482 \frac{M_a}{M_b} \frac{(5 + M_b^2)^{\frac{3}{2}}}{5 + M_a^2} \frac{k}{\sin \theta} \frac{f(\xi_{\alpha})}{f(\xi_A)} \frac{f(\xi_{\alpha})}{f(\xi_A)},
\]

where \( M_b \) denotes approaching Mach number and \( \nu \) is Prandtl-Meyer angle which is given by

\[
\nu = \nu(M) = \sqrt{\frac{r+1}{r-1}} \tan^{-1} \sqrt{\frac{r-1}{r+1} (M^2 - 1)} - \tan^{-1} \sqrt{M^2 - 1}.
\]

Moreover, from Eq. (4.9), the flow deflection angle, \( \theta \), through shock can be expressed as

\[
\alpha = \nu(M_a) - \nu(M_b) = \nu_b - \nu_a
\]

\[
\therefore \quad \theta = \nu_2 - \nu_a,
\]

where subscript \( a \) indicates the condition of the inviscid flow on the parallel surface of the airfoil. As is seen in Eq. (4.10), \( \theta \) does not depend on \( M_b \) and, consequently, the boat-tail angle \( \alpha \). This result indicates that the last fraction in right-hand side of Eq. (4.8) is a function of \( M_a \) only and, therefore, Eq. (4.8) can be reexpressed as

\[
\left( \frac{c}{h} \right)_{\text{Re}^{\alpha=\alpha}} = 0.1482 \frac{M_a}{M_b} \frac{(5 + M_b^2)^{\frac{3}{2}}}{5 + M_a^2} \frac{k}{\sin \theta} \frac{f(\xi_{\alpha})}{f(\xi_A)} \cdot \phi,
\]

where

\[
\phi = \frac{M_a}{M_b} \frac{(5 + M_b^2)^{\frac{3}{2}}}{5 + M_a^2},
\]

Comparing Eq. (4.11) with Eq. (3.13), it is found that \( \phi \) in Eq. (4.11) indicates the contribution of boat-tail angle to the magnitude of Reynolds number shape parameter, when \( M_a \) and \( M_b \) are fixed. If the boat-tail angle is zero, \( M_b \) is equal to \( M_a \) and, hence, \( \phi = 1 \). Since, as is seen in Eq. (4.12), \( \phi \) increases with increasing \( M_b \), the value of Reynolds number shape parameter increases with increasing boat-tail angle for a fixed \( M_a \). This result indicates that, if the free stream Mach number and Reynolds number shape parameter are fixed, the base pressure decreases with increasing boat-tail angle.

As has been just discussed above, the base pressure depends upon the boat-tail angle. However, if the boat-tail angle is assumed to be small, the approaching
Mach number $M_a$ is approximately equal to $M_a$ and, therefore, $\phi$ becomes unity approximately. This result leads to the statement that the effect of boat-tail angle has little contribution to the base pressure, if the boat-tail angle is small. And this has already been confirmed by experiment made by Chapman, Wimbrow and Kester.

5. Numerical Computation and Comparison with Experiment

As has been discussed in the last section, Eq. (4.11) gives the solution of base pressure on two-dimensional blunt-trailing edge wings at supersonic velocities, when the approaching boundary layer is laminar. However, it must be noted that the mixing coefficient $k$ in Eq. (4.11) still remains unknown. This arises from the fact that the turbulent flow theory does not give any quantitative result of eddy kinematic viscosity. In this circumstance the mixing coefficient must be left to be an empirical constant in the present approach. From the experimental results obtained by Schubauer and Klebanoff [6], Crocco and Lees used $k=0.03$ in their numerical calculation irrespective of the approaching Mach number. Hence, $k=0.03$ is also used in the present calculation. Moreover, if the airfoil is assumed to be thin, $M_a$ can be taken to be approximately equal to $M_\infty$.

Since, as is seen in Eq. (4.11), it seems to be much laborious to solve the equation with respect to $M_\infty$, on the assumption just mentioned above the numerical computation was carried out for free stream Mach numbers of 1.5, 2.0 and 3.1 by the following procedure: the free stream Mach number (which is assumed to be equal to $M_a$) and the boat-tail angle are first fixed. Then, the approaching Mach number $M_\infty$ is obtained by the use of Prandtl-Meyer relation and, consequently, $\phi$ can be calculated. Second, assume $M_\infty$, then the turning angle of the inviscid flow through Prandtl-Meyer fan at the trailing edge is obtained and, hence, the flow deflection angle, $\theta$, through shock is found, since it is equal to the resultant turning angle of the inviscid flow through Prandtl-Meyer expansion from $M_\infty$ to $M_\infty$. With these values and also from Korst's criterion $u_\infty^2 \leq$ is obtained and, consequently, $f(\xi_{\infty})$. Therefore, the Reynolds number shape parameter can be calculated for the given values of $M_\infty$, $\alpha$ and $M_\infty$.

In Figs. 3(a), 3(b) and 3(c) are presented the variations of base pressure ratio defined by $p_b/p_\infty$ with Reynolds number shape parameter for free stream Mach numbers of 1.5, 2.0 and 3.1, respectively, in which the boat-tail angle is included as a parameter. As is seen in the figures, the base pressure increases with increasing Reynolds number shape parameter. This result is reexpressed by the statement that the base pressure decreases with increasing Reynolds number. The numerical results also clearly indicate that the base pressure decreases with increasing boat-tail angle. Moreover, it is remarkable that the boat-tail angle has less effect on the base pressure for lower free stream Mach numbers.

Figs. 4(a), 4(b) and 4(c) show the variations of base pressure ratio with Reynolds number shape parameter at zero boat-tail angle for free stream Mach number of 1.5 together with experimental data obtained by Chapman, Wimbrow and Kester.
[5] for comparison. In the figures the first number in the representation of the airfoil model indicates thickness-chord ratio and the second number denotes the ratio of the height of the base to the maximum thickness, respectively. The boat-tail angle of each model falls in the range from $-2.87^\circ$ to $5.00^\circ$. Since
the base pressure is less influenced by the small boat-tail angle, as is seen in Fig. 3(a), the present theory at zero boat-tail angle gives a fairly good agreement with experimental data over a wide range of Reynolds number shape parameter. For the two wings with thin trailing edges (wings 0.05-0.25 and 0.05-0.50),
however, the base pressure is much lower at certain Reynolds numbers than it would be expected on the basis of the average value for the other wings. The base pressure data for wings 0.05–0.25 and 0.05–0.50 conform to the main data

![Figure 5(a)](image)

**Figure 5(a).** Comparison of the present theory with experiment. 
$M_{\infty}=2.0$, $t/c=0.10$, laminar boundary layer.

![Figure 5(b)](image)

**Figure 5(b).** Comparison of the present theory with experiment. 
$M_{\infty}=2.0$, $t/c=0.075$, laminar boundary layer.

![Figure 5(c)](image)

**Figure 5(c).** Comparison of the present theory with experiment. 
$M_{\infty}=2.0$, $t/c=0.05$, laminar boundary layer.
only at Reynolds number shape parameter greater than 0.12 and 0.05, respectively, as is seen in Fig. 4(c). Chapman and others pointed out from their test results that, if the data were taken in the order of increasing Reynolds numbers, the base pressure measurements on wing 0.05–0.25 repeated reasonably well, while the measurements sometimes failed to repeat, if the data were taken in the order of decreasing Reynolds numbers. They further remarked that the measurements on other wings and the measurements at higher Mach numbers could be repeated satisfactorily. Under these circumstances they tried to give an explanation to these non-conforming data as follows: although the reason for the instability to repeat measurements is not known as yet, one explanation that immediately suggests itself is that transition from the laminar to the turbulent flow occurred in the boundary layer in this Reynolds number range.

![Figure 6. Comparison of the present theory with experiment.](image)

$M_{in}$ = 3.1, laminar boundary layer.

In Figs. 5(a), 5(b) and 5(c) and in Fig. 6 are presented the variations of base pressure ratio with Reynolds number shape parameter at zero boat-tail angle for free stream Mach numbers of 2.0 and 3.1, respectively, together with Chapman’s data for comparison. In these cases of free stream Mach numbers, since the boat-tail angle of the airfoil model is very small, the agreement between theory and experiment is fairly good. Moreover, the experimental results shown in Figs. 4, 5 and 6 seem to indicate a tendency that the base pressure decreases with increasing boat-tail angle. On this point the present theory is also verified by the experiment.

Fig. 7 shows a comparison of the base pressure coefficient at zero boat-tail angle and at free stream Mach number of 2.0 between the present theory and Chapman’s another data [7]. In this case the quantitative agreement is also found between the theory and the experiment.

It must, however, be noted that the base pressure ratio has a critical value for decreasing Reynolds number shape parameter. Since the flow deflection angle through single oblique shock wave has a maximum value, for the given $M_2$, beyond which the inviscid flow solution cannot exist, the turning angle of the inviscid flow through Prandtl-Meyer expansion cannot increase beyond this
Figure 7. Comparison of the present theory with experiment. $M_\infty = 2.0$, $t/c = 0.075$, laminar boundary layer.

Figure 8. Variation of critical Reynolds number shape parameter with Mach number. Laminar boundary layer.

maximum value. This restriction, as pointed out by Chapman, defines a limiting value of possible solution of the base pressure and, hence, the critical value of Reynolds number shape parameter.

In Fig. 8 is presented the variation of critical Reynolds number shape parameter with free stream Mach number together with the limiting value of possible base pressure coefficient calculated by Chapman [1]. As is seen in the figure, the critical Reynolds number shape parameter decreases with increasing free stream Mach number for a given airfoil. It must, however, be noted that the critical Reynolds number shape parameter indicates an upper limit of the Reynolds number where the base pressure is influenced by the viscous effect for the given airfoil. This suggests that, for larger Reynolds number beyond the critical,
the base pressure may be kept constant at its limiting value, as has been already pointed out by Chapman.

6. CONCLUSION

Interaction between dissipative flow and neighbouring nearly isentropic stream accompanied with the flow separation was analysed and the results were applied to the base pressure on two-dimensional blunt-trailing edge wings at supersonic velocities and with laminar approaching boundary layer. In the present approach the interrelation between the viscous region and the external inviscid flow was obtained in a mathematically simpler form and Chapman's correlation between the base pressure and the ratio of boundary layer thickness at the base to the base diameter was clarified quantitatively. It was revealed that the mechanism of turbulent jet mixing plays the main role in determining the base pressure.

Numerical computation of the theoretical base pressure was carried out for free stream Mach numbers of 1.5, 2.0 and 3.1 under the assumption of $k=0.03$ irrespective of the Mach number and quantitative agreement was found between the present theory and the base pressure data of Chapman, Wimbrow and Kester on supersonic airfoils with blunt-trailing edgs. It has been found that the base pressure decreases with increasing boat-tail angle and its effect becomes smaller for lower free stream Mach numbers.

For the fixed free stream Mach number there is a critical value of Reynolds number shape parameter, below which the theoretical solution of the base pressure does not exist. In the range of Reynolds number shape parameter beyond its critical value, the base pressure is seriously influenced by the viscous effect and continues to decrease with increasing Reynolds number.

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