Solution of Nonparametric Shape Optimization Problems

by

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ABSTRACT

Structural optimization problems, in which sizes or CAD data are chosen as design variables, are called parametric structural optimization problems, while optimization problems of domains, in which linear elastic continua, flow fields, etc., are defined and functions describing the domain variations, such as mappings, are chosen as design variables, are called nonparametric structural optimization problems. For the nonparametric structural optimization problems, the optimization theory was formulated by expressing domain variation with a one parameter family of mappings defined in an initial domain. Using this theory, a derivative of an objective functional with respect to domain variation can be derived rigorously. It is known, however, that ordinary domain optimization problems lack sufficient regularity. This paper presents a regularization technique that we call the traction method. This technique is based on the idea of a gradient method in Hilbert space, which was shown by Cea. Starting with the linear form with respect to the domain variation that is given by the derivative of the mapping family and called the velocity, Cea demonstrated the use of a coercive bilinear form in Hilbert space to determine the velocity that minimizes the objective functional. Our proposal is to use the bilinear form that is defined for variational strain energy in an elastic continuum problem as an explicit form of the coercive bilinear form. The governing equation of the velocity indicates that we can determine the velocity as a displacement of the pseudo-elastic body defined in the design domain by loading a pseudo-external force in proportion to the negative value of the shape gradient function under constraints on the displacement of the invariable boundaries. We call this solution the traction method. The traction method is coupled with topological optimization methods, such as the technique using micro-scale voids. The validity of the proposed method is demonstrated by numerical analyses of linear elastic continua and flow fields.

1. Introduction

In classical mechanics, equilibrium and eigenvalue conditions of linear elastic continua, flow fields, magnetic fields, etc., are given by elliptic differential equations defined in terms of domains and boundary conditions. When design variables are considered as the geometrical shapes of the domains, the problems are called geometrical domain or nonparametric shape optimization problems that are distinguished from parametric structural optimization problems in which sizes or CAD data are chosen as design variables.

The theoretical basis concerning derivation of sensitivity functions, which we call shape gradient functions, to geometrical domain variation has been studied from early in this century. In 1908, Hadamard(1) showed the differentiability of variations of a geometrical domain with a smooth boundary in which an elliptic boundary value problem is defined.(2) Zolésio(3) extended the theory to domains with piecewise smooth boundaries. He formulated domain variation with a smooth transformation, or mapping, of Euclidean space just the same as the original domain. He called the domain variation the velocity field and this approach the material derivative method. The applicability of the material derivative method to engineering problems was demonstrated by Haug, Choi and Komkov.(4)

Although the shape gradient functions can be evaluated rigorously, the essential difficulty of domain optimization problems is the lack of regularity. Imari(5) pointed out an oscillation phenomena of boundary by moving nodes in a finite element mesh in proportion with sensitivity. A numerical analysis showing the necessity of the Lipschitz condition can be found in a monograph published by Haslinger and Neittaanmäki.(6) Braibant and Fleury(7) presented numerical results that indicated unrealistic shapes were generated by moving nodes in a finite element mesh. Based on that observation, they proposed the use of B-spline curves to control shapes.

Following the works of these pioneers, the author proposed a regularization technique which we call the traction method.(8),(9) This technique is based on the idea of a gradient method in Hilbert space, which was shown by Cea.(10) Starting with the linear form with respect to the velocity, Cea demonstrated the use of a coercive bilinear form in Hilbert space to determine the velocity that minimizes the objective functional. Our proposal is to use the bilinear form that is defined for variational strain energy in an elastic continuum problem as an explicit form of the coercive bilinear form. The governing equation of the velocity indicates that we can determine the velocity as a displacement of the pseudo-elastic body defined in the design domain by loading a pseudo-external force in proportion to the negative value of the shape gradient function under constraints on the displacement of the invariable boundaries. We call this solution the traction method because of this procedure. To conduct a numerical analysis, we can use any technique applicable to linear elastic problems, such as the finite element method or boundary element method. The irregularity of the domain optimization problems is confirmed by the authors(11) through a discussion of the ill-posedness that occurs when the gradient method in Hilbert space is applied directly. Introducing an idea to restrict the Hilbert space to a smoother one, a smoothing gradient method in Hilbert space is proposed. It is conclusively shown that a numerical method based on this idea coincides with the traction method.

This paper briefly describes the derivation of the shape gradient functions and the procedure of the traction method and presents numerical results for elas-

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2. Domain Variation

Let an open set $\Omega \subset \Omega_{\text{limit}} \in \mathbb{R}^n$, $n = 2, 3$ be an initial domain with a boundary $\Gamma$ and varied to a domain $\Omega_s \subset \Omega_{\text{limit}}$ with a boundary $\Gamma_s$. Assuming $\tilde{T}_s(\tilde{x})$, $s \geq 0$, $\tilde{x} \in \tilde{\Omega}_s$, as a one parameter family which is mapped from the initial closed domain $\Omega$ to the varied closed domain $\tilde{\Omega}_s$ as shown in Fig. 1, i.e. $\tilde{T}_s(\tilde{x})$: $\Omega \ni \tilde{x} \rightarrow \tilde{x} \in \tilde{\Omega}_s$, we can rewrite it by the ordinary differential equation with respect to $s$:

$$\tilde{T}_s(\tilde{x}) = \Omega_s, \quad \tilde{\mathbf{V}}(\tilde{\Omega}_s) = \tilde{\mathbf{V}}(\tilde{\Omega}) \in \mathbb{D}, \quad s > 0$$ (1)

In this paper, for $n$ dimensional vectors the notation $(\cdot)$ and the tensor notation with subscripts are used.

Considering $s$ as a time history, we call $\tilde{\mathbf{V}}$ the velocity function or velocity field. Assuming a restriction of domain variation on the subdomain or subboundary $\Theta$ of the design domain, the kinematically admissible set of the velocity field $\tilde{\mathbf{V}}$ is given by

$$D = \{\tilde{\mathbf{V}} \in (C^1(\tilde{\Omega}_s))^n \mid \tilde{\mathbf{V}}(\tilde{x}) = \tilde{0}, \quad \tilde{x} \in \Theta\}$$ (2)

For the sake of simplicity, we assume the measure of $\Theta$ is not zero.

When a domain functional $J_{\Omega_s}$ and a boundary functional $J_{\Gamma_s}$ of a distributed function $\phi_s$, 

$$J_{\Omega_s} = \int_{\tilde{\Omega}_s} \phi_s \, dx$$ (3) 

$$J_{\Gamma_s} = \int_{\Gamma_s} \phi_s \, d\Gamma$$ (4)

are considered, their derivatives with respect to $s$ are given by

$$J_{\Omega_s} = \int_{\tilde{\Omega}_s} \phi'_s \, dx + \int_{\tilde{\Gamma}_s} \phi_s n^T \tilde{\mathbf{V}} \, d\Gamma$$ (5) 

$$J_{\Gamma_s} = \int_{\Gamma_s} \{ \phi'_s + (\phi_s n_i + \phi_s \kappa) n^T \tilde{\mathbf{V}} \} \, d\Gamma$$ (6)

where $n$ is an outward unit normal vector and $(\cdot)^T$ denotes the transpose. The notation $\kappa$ denotes the mean curvature. The shape derivative $\phi'_s$ of the distributed function $\phi_s$ indicates the derivatives under a spatially fixed condition. In the tensor notation, the Einstein summation convention and the gradient notation $(\cdot)_i = \partial(\cdot)/\partial x_i$ are used.

3. Shape Gradient Function

Many basic domain optimization problems can be formulated in terms of the finding the domain $\tilde{\Omega}_s \subset \tilde{\Omega}_{\text{limit}}$ that minimize an objective functional $F(\tilde{\mathbf{u}})$ of a state variable function $\tilde{\mathbf{u}}$ subject to a variational form of a state equation $g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = 0$, $\forall \tilde{\mathbf{w}} \in \mathbb{W}$, under a constraint on the measure of the domain:

Given $\tilde{\mathbf{u}}$, $M$, and coefficients in $g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$, appropriately smooth and fixed in $\tilde{\Omega}_{\text{limit}}$, 

$$F(\tilde{\mathbf{u}}), \quad \tilde{\mathbf{u}} \in U$$ (8)

that minimize $F(\tilde{\mathbf{u}})$, $\tilde{\mathbf{u}} \in U$

subject to $g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) = 0$, $\forall \tilde{\mathbf{w}} \in \mathbb{W}$

$$\text{meas}(\tilde{\Omega}_s) = \int_{\tilde{\Omega}_s} \, dx \leq M$$ (11)

where $U$ and $W$ are the admissible set of the state function $\tilde{\mathbf{u}}$ and the variational state function, or adjoint function, $\tilde{\mathbf{w}}$ respectively.

By applying the Lagrange multiplier method, or adjoint method, and the formulae of derivatives of functionals with respect to domain variation of the velocity field (8) the Lagrange functional $L(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{A}}, \tilde{T}_s)$ and its derivative to $s$, $\tilde{\mathbf{L}}$, are obtained as follows.

$$L = F(\tilde{\mathbf{u}}) + g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}) + A(\text{meas}(\tilde{\Omega}_s) - M)$$ (12) 

$$\tilde{\mathbf{L}} = g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}') + g(\tilde{\mathbf{u}}', \tilde{\mathbf{w}}) + A(\text{meas}(\tilde{\Omega}_s) - M)$$

$$+ l_G(\tilde{\mathbf{V}})$$ (13)

where $\tilde{\mathbf{w}}'$ and $\tilde{\mathbf{A}}$ are the Lagrange multipliers of the state equation and measure constraint respectively. When $\tilde{\mathbf{u}}$ is determined by the state equation:

$$g(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}') = 0, \quad \forall \tilde{\mathbf{w}}' \in \mathbb{W}$$ (14)

$\tilde{\mathbf{w}}'$ by the adjoint equation:

$$F(\tilde{\mathbf{u}}') + g(\tilde{\mathbf{u}}', \tilde{\mathbf{w}}) = 0, \quad \forall \tilde{\mathbf{w}}' \in \mathbb{W}$$ (15)

and $A \geq 0$ by

$$A(\text{meas}(\tilde{\Omega}_s) - M) = 0$$ (16) 

$$\text{meas}(\tilde{\Omega}_s) \leq M$$ (17)

Eq. (13) becomes

$$\tilde{\mathbf{L}}_{\text{u,v}, A} = \tilde{\mathbf{F}}(\tilde{\mathbf{u}})_{\text{u,v}, A} = l_G(\tilde{\mathbf{V}})$$

$$= \int_{\Gamma_s} \tilde{\mathbf{G}}_s(\tilde{x}) \tilde{\mathbf{V}} \, dx = \int_{\Gamma_s} \tilde{\mathbf{G}}_s \tilde{\mathbf{N}} \tilde{\mathbf{V}} \, dx$$ (18)

The vector function $\tilde{\mathbf{G}}_s(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{A}}, \tilde{T}_s)$ has the meaning of a sensitivity function to the velocity field $\tilde{\mathbf{V}}$ that we call the shape gradient function. The scalar function $G(\tilde{\mathbf{u}}, \tilde{\mathbf{w}}, \tilde{\mathbf{A}}, \tilde{T}_s)$ is called the shape gradient density function.

4. Traction Method

The traction method has been proposed as a procedure for solving the velocity field $\tilde{\mathbf{V}} \in \mathbb{D}$ by

$$a(\tilde{\mathbf{V}}, \tilde{\mathbf{w}}) = -l_G(\tilde{\mathbf{w}}), \quad \forall \tilde{\mathbf{w}} \in \mathbb{D}$$ (19)
\[ a(\bar{u}, \bar{v}) = \int_{\Omega_s} C_{ijkl} u_{k,i} v_{l,j} \, dx. \]  \hspace{1cm} (20)

where \( l_G(\cdot) \) is the linear form defined by Eq. (18).

When the shape gradient density function \( G \) has appropriate smoothness and the measure of the restriction domain or boundary \( \Theta \) is not zero, the velocity fields \( \tilde{V} \) can be determined by Eq. (19). That the solution \( \tilde{V} \) decrease the objective functional in convex problems is assured by using the coerciveness of the bilinear form \( a(\cdot, \cdot) \). (8)

Equation (19) indicates that the velocity fields decreasing the objective functional are obtained as a displacement of the pseudo-elastic body defined in \( \Omega_s \) by the loading of the pseudo-external force in proportion to \(-\tilde{G}_s\) under constraints on displacement of the invariable boundaries as shown in Fig. 2.

In this paper, FEM was employed to find the solution of Eq. (19).

The Lagrange multiplier \( \Lambda \) that satisfies Eqs. (16) and (17) is determined as follows. Since \( \Lambda \) contributes to the pseudo force \(-\tilde{G}_s\) as a uniform boundary force, the relation among the variation of the uniform boundary force \( \Delta \Lambda \tilde{u} \), the variation of the velocity \( \Delta \tilde{V} \) and the variation of the measure of the domain \( \Delta \text{meas}(\Omega_s) \) is obtained by elastic deformation analysis based on the following equation loaded with the uniform boundary force \( \Delta \Lambda \tilde{u} \) as shown in Fig. 3.

\[ a(\Delta \tilde{V}, \tilde{w}) = \Delta \Lambda \int_{\Omega_s} \tilde{n}^T \tilde{w} \, d\Gamma, \quad \forall \tilde{w} \in D \]  \hspace{1cm} (21)

\[ \Delta \text{meas}(\Omega_s) = \int_{\Gamma_s} \tilde{n}^T \Delta \tilde{V} \, d\Gamma \]  \hspace{1cm} (22)

The procedure of the traction method can be described as follows.

1) Start with a state function analysis followed by an adjoint function analysis, if necessary, depending on the problem to be solved.

2) Using the results, calculate the shape gradient function on the design boundary.

3) Using the shape gradient function, analyze \( \tilde{V} \) by Eq. (19).

4) Deform the domain with \( \tilde{V} \) and evaluate the domain measure.

5) Determine \( \Lambda \) that satisfies Eqs. (16) and (17) using the results of Eqs. (21) and (22).

6) Multiplying the velocity function by an incremental value of \( \delta \), update the domain and return to step (1).

7) Terminate the procedure based on the results of the state function analysis.

5. Domain Optimization Problems

For some basic domain optimization problems in engineering, we can derive the shape gradient functions in the following manner.

5.1 Mean Compliance Minimization Problem Let \( \Omega \) be a domain of a linear elastic continuum loaded with a volume force \( \tilde{f} \) in \( \Omega \) and a traction \( \tilde{P} \) on a boundary \( \Gamma_1 \) under a constraint on the displacement of a boundary \( \Gamma_0 \) as shown in Fig. 4.

A simple minimization problem of mean compliance by domain variation on a boundary \( \Gamma_\text{design} \subset \Gamma_s \) under a constraint on the volume of the domain is formulated as follows.

\[ \text{Given} \quad \Omega, \ M, \ \tilde{P}, \ \tilde{f}, \quad \text{and} \quad C_{ijkl}, \ i, j, k, l = 1, 2, \cdots, n, \]  \hspace{1cm} (23)

appropriately smooth and fixed in \( \Omega_\text{limit} \).

\[ \text{find} \quad \Omega_s = \tilde{T}_s(\Omega), \]  \hspace{1cm} (24)

\[ \tilde{T}_s(\Omega) = \tilde{V}(\Omega_s) \in D, \quad s \geq 0 \]

that minimize \( l(\tilde{u}), \quad \tilde{u} \in U \)  \hspace{1cm} (25)

subject to \( a(\tilde{u}, \tilde{w}) = l(\tilde{w}), \quad \forall \tilde{w} \in U \)

\[ \text{meas}(\Omega_s) = \int_{\Omega_s} dx \leq M \]  \hspace{1cm} (26)

\[ \text{where the bilinear form} \quad a(\cdot, \cdot) \quad \text{is defined by Eq. (20),} \]

the linear form \( l(\cdot) \) is defined by

\[ l(\tilde{w}) = \int_{\Omega_s} f_i w_i \, dx + \int_{\Gamma_1} P_i w_i \, d\Gamma \]  \hspace{1cm} (28)

and \( U \) is the set of the admissible displacements satisfying \( \tilde{w} = 0 \) on \( \Gamma_0 \).

For this problem, the shape gradient function is derived by using the Lagrange multiplier method and
the formulae in Eqs. (5) and (6) as follows.\(^{(12)}\)

\[
G_s = -C_{ijkl}u_{k,i}u_{l,j} + 2f_iu_i + 2(P_{ijk}u_{i,n} + P_{lij}u_{j,n} + P_{lui}u_{i,n}) + \Lambda \tag{29}
\]

5.2 Moving Problem of Vibrational Eigenvalues
Let us consider a moving problem of vibrational eigenvalues \(\lambda_{(r_m)}, m = 1, 2, \cdots, N,\) of modal numbers \(r_m, m = 1, 2, \cdots, N,\) with eigen-modes \(\bar{u}_{(r_m)}, m = 1, 2, \cdots, N,\) to a specified direction, or weights, \(\alpha_{(r_m)}, m = 1, 2, \cdots, N:\)

\[
\text{Given } \Omega, M, r_m \text{ and } \alpha_{(r_m)}, m = 1, 2, \cdots, N, \quad C_{ijkl}, i, j, k, l = 1, 2, \cdots, n, \text{ density } \rho, \text{ appropriately smooth and fixed in } \Omega_{\text{limit}}, \tag{30}
\]

\[
\text{find } \Omega_s = T_s(\Omega), \quad \bar{T}_s(\Omega) = \bar{V}(\Omega_s) \subset D, \quad s \geq 0 \tag{31}
\]

\[
\text{that maximize } \sum_{m=1}^{N} \alpha_{(r_m)}\lambda_{(r_m)} \tag{32}
\]

\[
\text{subject to } a(\bar{u}_{(r_m)}, \bar{w}) = \lambda_{(r_m)}b(\bar{u}_{(r_m)}, \bar{w}), \quad \bar{u}_{(r_m)} \in U \forall \bar{w} \in U, \quad m = 1, 2, \cdots, N \tag{33}
\]

\[
\text{meas}(\Omega_s) = \int_{\Omega_s} dx \leq M \tag{34}
\]

where the bilinear form \(b(\cdot, \cdot)\) is defined by

\[
b(\bar{u}, \bar{w}) = \int_{\Omega_s} \rho u_i w_i dx \tag{35}
\]

The shape gradient function for this problem is derived as follows.

\[
G_s = \sum_{m=1}^{N} \alpha_{(r_m)}(-C_{ijkl}u_{k,i}u_{l,j} + \lambda_{(r_m)}\rho u_{(r_m)}u_{(r_m)} + \Lambda) \tag{36}
\]

5.3 Frequency Response Minimization Problems
By putting three frequency response functionals of strain energy, kinetic energy and absolute mean compliance on the objective functionals, the following optimization problems are formulated.

\[
\text{Given } \Omega, M, C_{ijkl}, i, j, k, l = 1, 2, \cdots, n, \rho, \text{ volume force } f \cos \omega t, \text{ and traction force } P \cos \omega t, \text{ appropriately smooth and fixed in } \Omega_{\text{limit}}, \tag{37}
\]

\[
\text{find } \Omega_s = T_s(\Omega), \quad \bar{T}_s(\Omega) = \bar{V}(\Omega_s) \subset D, \quad s \geq 0 \tag{38}
\]

\[
\text{that minimize } \frac{1}{2}a(\bar{u}, \bar{u}), \quad \frac{1}{2}b(\bar{u}, \bar{w}) + a(\bar{u}, \bar{w}) = \|l(\bar{u})\|, \quad \bar{u} \in U \tag{39}
\]

\[
\text{subject to } -\omega^2 b(\bar{u}, \bar{w}) + a(\bar{u}, \bar{w}) = l(\bar{w}), \quad \forall \bar{w} \in U \tag{40}
\]

\[
\text{meas}(\Omega_s) = \int_{\Omega_s} dx \leq M \tag{41}
\]

5.3.1 Strain Energy Minimization Problem
The strain gradient function for the strain energy minimization problem is derived as

\[
G_s = \frac{1}{2}C_{ijkl}u_{k,i}u_{l,j} - C_{ijkl}u_{k,i}w_{i,j} + \omega^2 \rho u_i w_i + \Lambda \tag{42}
\]

where the displacement amplitude \(\bar{u}\) and the adjoint displacement amplitude \(\bar{w}\) are calculated with the modal displacement \(\xi_{(m)}\) and the modal adjoint displacement \(\eta_{(m)}\) by

\[
\bar{u} = \sum_{m=1}^{\infty} \xi_{(m)} \bar{u}_{(m)}, \quad \xi_{(m)} = \frac{l(\bar{u}_{(m)})}{\lambda_{(m)} - \omega^2} \tag{43}
\]

\[
\bar{w} = \sum_{m=1}^{\infty} \eta_{(m)} \bar{u}_{(m)}, \quad \eta_{(m)} = \frac{\lambda_{(m)} l(\bar{u}_{(m)})}{(\lambda_{(m)} - \omega^2)^2} \tag{44}
\]

5.3.2 Kinetic Energy Minimization Problem
For the kinetic energy minimization problem, the shape gradient function is derived as:

\[
G_s = \frac{1}{2} \omega^2 \rho u_i u_i - C_{ijkl} u_{k,i} w_{i,j} + \omega^2 \rho u_i w_i + \Lambda \tag{45}
\]

where \(\bar{u}\) is are calculated by Eq. (43) and \(\bar{w}\) is calculated by

\[
\bar{w} = \sum_{m=1}^{\infty} \eta_{(m)} \bar{u}_{(m)}, \quad \eta_{(m)} = \frac{\omega^2 l(\bar{u}_{(m)})}{(\lambda_{(m)} - \omega^2)^2} \tag{46}
\]

5.3.3 Absolute Mean Compliance Minimization Problem
For the absolute mean compliance minimization problem, the shape gradient function is derived as:

\[
G_s = -C_{ijkl} u_{k,i} u_{l,j} + \omega^2 \rho u_i w_i + \Lambda \tag{47}
\]

where \(\bar{u}\) is are calculated by Eq. (43) and \(\bar{w}\) is calculated by

\[
\bar{w} = \begin{cases} 
\bar{u} & \text{if } (l(\bar{u})) \geq 0 \\
-\bar{u} & \text{if } (l(\bar{u})) < 0
\end{cases} \tag{48}
\]

5.4 Dissipation Energy Minimization Problem of Viscous Flow Field
Let \(\Omega\) be a flow field of an incompressible Newtonian fluid in a steady state as shown in Fig. 5. The fluid flows in from a boundary \(\Gamma_0\) and flows out from a boundary \(\Gamma_1\).

The minimization problem of dissipation energy by domain variation that occurs on a boundary \(\Gamma_{\text{design}} \subset \Gamma, \Gamma_{\text{design}} \cap \Gamma_0 = \emptyset, \Gamma_{\text{design}} \cap \Gamma_1 = \emptyset\) under a constraint
on the volume of the domain is formulated as follows.

\[
\text{Given } \Omega, M, \rho \text{ and viscous coefficient } \mu \text{ appropriately smooth and fixed in } \Omega_{\text{limit}},
\]

\[
\text{find } \Omega_s = \tilde{T}_s(\Omega),
\]

\[
\tilde{T}_s(\Omega) = \tilde{V}(\Omega_s) \in D, \quad s \geq 0
\]

that minimize

\[
a(\bar{u}, \bar{u}) + \tilde{a}(\bar{u}, \tilde{u}), \quad \tilde{u} \in U
\]

subject to

\[
b(\nabla \bar{u}^T, \tilde{w}) + a(\bar{u}, \tilde{w}) = \langle p, w_i \rangle,
\]

\[
\forall \tilde{w} \in W,
\]

\[
\langle q, u_i \rangle = 0, \quad \forall q \in Q
\]

\[
\text{meas}(\Omega_s) = \int_{\Omega_s} dx \leq M
\]

where Eqs. (52) and (53) are variational forms of the Navier-Stokes equation and the continuity equation. The bilinear form \( b(\cdot, \cdot) \) in the convective term is defined by Eq. (35) where \( \nabla \bar{u}^T \bar{u} \) is given as \( u_{i,j}u_j \) in the tensor notation. The bilinear form \( a(\cdot, \cdot) \) in the viscous term and \( \tilde{a}(\cdot, \cdot) \) in the dissipation energy are defined by Eq. (20) where \( C_{ijkl} = \mu \delta_{ik}\delta_{jl} \) and \( \tilde{C}_{ijkl} = \mu \delta_{ik}\delta_{jl} \) in \( a(\cdot, \cdot) \). The bilinear form \( \langle \cdot, \cdot \rangle \) in the pressure term is defined as

\[
\langle p, q \rangle = \int_{\Omega_s} pq \, dx
\]

The velocity \( \bar{u} \), the adjoint velocity \( \tilde{w} \), the pressure \( p \) and the adjoint pressure \( q \) are in the function spaces having appropriate smoothness, respectively,

\[
U = \{ \bar{u} \mid \bar{u} = \bar{u}_0 \text{ given on } \Gamma_0, \ u_{i,j}u_j = 0 \text{ on } \Gamma_1, \quad \bar{u} = \overline{0} \text{ on } \Gamma_0 \cup \Gamma_1 \}
\]

\[
W = \{ \tilde{w} \mid \tilde{w} = \overline{0} \text{ on } \Gamma \}\}
\]

\[
Q = \{ p \mid p = 0 \text{ on } \Gamma_1 \}
\]

Applying the Lagrange multiplier method, the shape gradient function for this problem is derived as follows.

\[
G = -\mu \bar{w}_{i,j}u_{i,j} + \mu u_{i,j}(u_{i,j} + u_{j,j}) + \lambda
\]

where the adjoint velocity \( \bar{w} \) is calculated by

\[
b(\nabla \bar{u}^T, \tilde{w}) + b(\nabla \bar{u}^T, \tilde{w}) + a(\bar{u}, \tilde{w})
\]

\[-2(a(\bar{u}, \tilde{w}^i) + \tilde{a}(\bar{u}, \bar{w}^i)) = \langle q, w_i \rangle, \quad \forall \tilde{w}^i \in U \]

\[
\langle q, w_i \rangle = 0, \quad \forall q \in Q
\]

\[
\text{6. Numerical Results}
\]

For the optimization problems provided with shape gradient functions, we can apply the traction method. Figure 6 and 7 show optimized shapes for the mean compliance minimization problems obtained by a shape analysis system developed with a general purpose FEM code.(22)

The result for the moving problem of vibrational eigenvalues is shown in Fig. 8.(13) This problem involved finding the maximized shape of \( \lambda_1 + \lambda_2 \) for a beam-like continuum clamped at both ends. For the frequency response minimization problems, we obtained various results depending on the objective functional as shown in Fig. 9.(13) We confirmed, however,

\[
\text{7. Coupling with Topological Optimization Method using Micro-Scale Voids}
\]

An attempt was also made to couple the traction method with the topological optimization method using micro-scale voids as proposed by Bendsoe and Kikuchi.(21) Their approach is based on the idea of periodic microstructures as shown in Fig. 11. They formulated a domain optimization problem, including a consideration of topology, to find the size and orientation functions of microstructures. They found the relationship between the design functions and macro properties of the material by using the theory of the homogenization method.

Figures 12 shows the process optimized for one part of an automotive suspension system with the objective of minimizing the mean compliance analyzed by a topology and shape analysis system developed with a general purpose FEM code. Figure 12 (a) shows the original design of this automotive part. We started with a two-dimensional problem. Assuming the design domain as shown in Fig. 12, we obtained the optimized topology by the topological optimization method. Based on the result, we made a three-dimensional model by selecting elements filled with the material. Applying the traction method to the three-dimensional model, we finally obtained the shape shown in Fig. 12.

\[
\text{8. Conclusion}
\]

This paper presented a numerical analysis technique called the traction method for application to geometrical shape optimization problems of domains in which boundary value problems and initial value problems of
partial differential equations, such as elastic continua and flow fields, are defined. For this type of optimization problems, difficulty had been found in solving them because of the lack of regularity. With this technique, most geometrical shape optimization problems can be solved keeping the same boundary smoothness as the initial domain. By coupling this technique with the topological optimization method, a more sophisticated system can be developed.

References


