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Geometric log Hodge structures on the standard log point

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ABSTRACT. We construct natural polarized log Hodge structures associated to a projective log deformation over the standard log point.

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1. Introduction

Since K. Kato and S. Usui introduced the notion of polarized log Hodge structure (see [15]), the theory rapidly grew. On the other hand, to apply the vast theory to geometry, evidently necessary are results of the type that a geometric family yields polarized log Hodge structures.

More precisely, let $f : P \rightarrow S$ be a projective log smooth morphism of fs log analytic spaces. Then, the question is whether f yields polarized log Hodge structures over S . The current situation concerning this question is summarized in Section 9 of [10] by T. Kajiwara, K. Kato, and the second author of this paper. Roughly speaking, the situation is as follows: Although

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there are satisfactory, affirmative answers in the case where the base S is log smooth, there are no nontrivial example when the base is not log smooth, even when the base is the standard log point.

The authors of [10] developed the theory of log Picard varieties and log Albanese varieties under the assumption that the answer to this question is affirmative (that f is *good* in their terminology), and stated that one can expect the goodness if f is exact. Note that the last condition that f is exact is always satisfied when S is the standard log point.

In this paper, we prove the following.

THEOREM 1.1 (= THEOREM 2.7). *A projective log deformation over the standard log point is good.*

That is, f is indeed good if S is the standard log point and if P is a projective log deformation which means that P is projective and is locally isomorphic to the special fiber of a semistable family with the natural log structure (this usage of “log deformation” was introduced by J. H. M. Steenbrink in [21]). Accordingly, the theory of [10] can be applied to such an f so that we now have for such an f the log Picard variety and the log Albanese variety, and so on.

For readers’ convenience, we include here a vague but more understandable statement.

COROLLARY 1.2. *Let $f : P \rightarrow S$ be a log deformation over the standard log point. Then the triple*

$$(R^q f_*^{\log} \mathbf{Z}, (R^q f_* \omega_{P/S}, F), \iota)$$

is a log Hodge structure of weight q for every integer q , where $f^{\log} : (P^{\log}, \mathcal{O}_P^{\log}) \rightarrow (S^{\log}, \mathcal{O}_S^{\log})$ is the associated morphism of ringed spaces, $\omega_{P/S}$ denotes the relative log de Rham complex, F the decreasing filtration induced by the stupid filtration on $\omega_{P/S}$, and $\iota : R^q f_^{\log} \mathbf{Z} \rightarrow \mathcal{O}_S^{\log} \otimes R^q f_* \omega_{P/S}$ the natural morphism induced by the canonical morphism $\mathbf{Z} \hookrightarrow \mathcal{O}_P^{\log} \rightarrow \omega_{P/S}^{\log}$.*

In Section 2, we review the definition of polarized log Hodge structure and the definition of the goodness and state our main theorem 2.7. In Section 3, we review the paper [5] which shows that a projective log deformation yields nilpotent orbits. Since a nilpotent orbit is an equivalent notion of a polarized log Hodge structure, this result in [5] already associates polarized log Hodge structures to a projective log deformation. However, the construction in [5] heavily depends on the various double complex constructions à la Steenbrink, and quite far from the definition of the goodness in [10] (which does not use any double complex). Thus, as is explained in Section 4, our task is to show that the two constructions in [5] and in [10] coincide. To see this, we introduce

an intermediate one, and compare it with each of two one by one. In Section 5, we prove the coincidence of the construction of [5] including double complexes with the intermediate one. There, after recalling several definitions and results in [5], we prove the commutativity of the diagram as expected. In Section 6, we give several isomorphisms concerning log Riemann–Hilbert correspondences, which is essentially a part of the theory in [7]. In Section 7, we prove the coincidence of the construction of [10] with the intermediate one. A key is a comparison between the relative log de Rham complex and the absolute one. In Section 8, by using the results obtained so far, we complete the proof of the main theorem. Finally, in Section 9, we apply the geometric machinery of [10], and define the log Picard and log Albanese varieties for a projective log deformation and state some results.

We note that it is well-known that the answer to the original question is affirmative for the non-log case, that is, the case of usual analytic spaces. This is classical but we could not find an appropriate reference, so we include a sketch of the proof as an appendix.

NOTATION AND TERMINOLOGY. For the basic notions on log geometry, see [15]. Here we review some of them. A monoid means a commutative semigroup. For an analytic space S , we denote by $\mathcal{O}_S$ the sheaf of holomorphic functions over S . A *log structure* on S is a pair of a sheaf of monoids M and a homomorphism $\alpha: M \rightarrow \mathcal{O}_S$ of sheaves of monoids such that the induced homomorphism $\alpha^{-1}(\mathcal{O}_S^\times) \rightarrow \mathcal{O}_S^\times$ is an isomorphism ([15] 2.1.1). Here we regard $\mathcal{O}_S$ as a sheaf of monoids by multiplication. Let P be a monoid. Then P^{gp} is an abelian group that represents the functor $A \mapsto \text{Hom}(P, A)$ (the set of the homomorphisms from P to an abelian group A as monoids). A monoid P is said to be *fs* (fine and saturated) if it is finitely generated, if the canonical homomorphism $P \rightarrow P^{\text{gp}}$ is injective and if for any $a \in P^{\text{gp}}$, a belongs to the image of P whenever there is an $n \geq 1$ such that a^n belongs to the image of P ([15] 2.1.4). A log structure M on an analytic space S is *fs* if, locally on S , there exist an fs monoid P and a homomorphism $\beta: P_S \rightarrow \mathcal{O}_S$ of sheaves of monoids such that the pushout of $P_S \leftarrow \beta^{-1}(\mathcal{O}_S^\times) \rightarrow \mathcal{O}_S^\times$ endowed with the induced homomorphism to $\mathcal{O}_S$ is isomorphic to M as a log structure ([15] 2.1.5). An *fs log analytic space* is a pair of an analytic space and an fs log structure on it ([15] 2.1.5). For an fs log analytic space S , we denote by M_S the log structure on it. A morphism of fs log analytic spaces is called *proper* (resp. *separated*) if its underlying continuous map of topological spaces is so (cf. [15] 0.7.5). See 2.1.11 of [15] for the definition of log smoothness of a morphism of fs log analytic spaces. There is a natural functor from the category of fs log analytic spaces to that of ringed spaces: $S \mapsto (S^{\text{log}}, \mathcal{O}_S^{\text{log}})$. See [15] 2.2.3–2.2.4.

Throughout the paper, we fix a standard log point

$$0 := (\mathrm{Spec} \mathbf{C}, \mathbf{N} \oplus \mathbf{C}^\times),$$

which is an fs log analytic space. Here the structural homomorphism $M_0 = \mathbf{N} \oplus \mathbf{C}^\times \rightarrow \mathcal{O}_0$ sends the generator 1 of $\mathbf{N}$ to zero. This 1 as a section of M_0 is denoted by t . The topological space 0^{log} is canonically identified with the unit circle $\{\alpha \in \mathbf{C} \mid \alpha\bar{\alpha} = 1\}$ by definition. Let $\log t$ be a section of the sheaf of logarithms $\mathcal{L}_0 \subset \mathcal{O}_0^{\mathrm{log}}$ such that $\exp(\log t) = t \in M_0^{\mathrm{gp}}$, which is determined modulo $2\pi i\mathbf{Z}$. We often identify a sheaf of abelian groups on 0 with the module of the global sections of it.

For simplicity, we often denote the pullback of a sheaf on a space by the same symbol as that for the original sheaf.

Sometimes, we omit the dot in the symbol $A^\cdot$ for a complex. For example, $\omega_{Y/0}$ denotes the relative log de Rham complex $\omega_{Y/0}^\cdot$.

2. Main theorem

In this section, we state our main theorem 2.7. For this, we recall the definition of polarized log Hodge structure and the definition of the goodness only in the case where the base is the standard log point 0 , which are enough for our purpose. See [9, 2.3], [15, 2.4], and [10] for the general theories.

2.1. Let q be an integer. A *log Hodge structure* of weight q over the standard log point 0 is a triple $(H_{\mathbf{Z}}, H_{\mathcal{O}}, \iota)$, where $H_{\mathbf{Z}}$ is a locally free sheaf of finitely generated $\mathbf{Z}$ -modules over 0^{log} , $H_{\mathcal{O}}$ is a $\mathbf{C}$ -vector space regarded as a sheaf of $\mathcal{O}_0$ -modules endowed with a decreasing filtration $F^\cdot$, and ι is an isomorphism

$$\iota : \mathcal{O}_0^{\mathrm{log}} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \cong \mathcal{O}_0^{\mathrm{log}} \otimes_{\mathcal{O}_0} H_{\mathcal{O}}$$

of $\mathcal{O}_0^{\mathrm{log}}$ -modules satisfying the following conditions (i)–(iii). Let z be the point of 0^{log} corresponding to 1, where we identify the set 0^{log} with the set $\{\alpha \in \mathbf{C} \mid \alpha\bar{\alpha} = 1\}$.

(i) For any sufficiently shifted specialization $a : \mathcal{O}_{0,z}^{\mathrm{log}} \rightarrow \mathbf{C}$ (which means that $\exp(a(\log t))$ is sufficiently near to $0 \in \mathbf{C}$), the specialization $\mathbf{C} \otimes_{\mathcal{O}_{0,z}^{\mathrm{log}}} \iota$ gives a Hodge structure of weight q in the usual sense.

(ii) Any specialization $\mathbf{C} \otimes_{\mathcal{O}_{0,z}^{\mathrm{log}}} \iota$ together with the monodromy filtration $W(N)[-q]$, where N is the monodromy, is a mixed Hodge structure in the usual sense.

(iii) Griffiths transversality is satisfied, that is,

$$V(F^p) \subset \omega_0^1 \otimes_{\mathcal{O}_0} F^{p-1}$$

for all $p \in \mathbf{Z}$, where $\nabla : H_{\mathcal{O}} \rightarrow \omega_0^1 \otimes_{\mathcal{O}_0} H_{\mathcal{O}}$ is induced from $d : \mathcal{O}_0^{\log} \rightarrow \omega_0^1 \otimes_{\mathcal{O}_0} \mathcal{O}_0^{\log}$.

We explain that the general definition is equivalent to the above definition in this case. Since we consider only the pure log Hodge structure, the admissibility ([9, 2.3.5 (i)]) is an empty condition. As for the condition [9, 2.3.5 (ii)], the monodromy cone (denoted by “ $C(s)$ ” there) is isomorphic to $\mathbf{R}_{\geq 0}$, and has two faces. The part of [9, 2.3.5 (ii)] where “ σ ” there is trivial is equivalent to the above condition (i). The part of [9, 2.3.5 (ii)] where “ σ ” is the whole monodromy cone is equivalent to the above condition (ii) because for any two specializations a, b , the data $\iota \otimes_{\mathcal{O}_{0,z}^{\log}, a} \mathbf{C}$ with $W(N)[-q]$ is a mixed Hodge structure if and only if the data $\iota \otimes_{\mathcal{O}_{0,z}^{\log}, b} \mathbf{C}$ with $W(N)[-q]$ is a mixed Hodge structure (hence, (ii) is even equivalent to the condition that “There is a specialization...”). Finally, the condition [9, 2.3.5 (iii)] is equivalent to the above condition (iii).

REMARK 2.2. The authors learned from K. Kato that (i) in the above definition is in fact a consequence of the other conditions. (Though we do not use this fact, we note that it makes the last sentence of 8.2 in the proof of the main theorem unnecessary.) We sketch the proof here. It is enough to prove the following:

Let N be a nilpotent endomorphism on a finite dimensional $\mathbf{R}$ -vector space H and let F be a decreasing filtration on $H_{\mathbf{C}}$. Let $(W(N)[-q], F)$ be the mixed Hodge structure. Then, $\exp(iyN)F$ is a Hodge structure of weight q for any sufficiently large $y > 0$.

To see this, let $\hat{F}$ be the $\mathbf{R}$ -split mixed Hodge structure associated with $(W(N)[-q], F)$ of Cattani–Kaplan–Schmid [2] (cf. [15, 6.1.2]). Let ν be the splitting of $\hat{F}$. Then we have $\text{Ad}(\nu(t))N = t^{-2}N$ for any $t \in \mathbf{R}_{>0}$ and $\nu(t)F \rightarrow \hat{F}$ ($t \rightarrow \infty$) ([15, Lemma 6.1.11 (ii)]) so that $\nu(\sqrt{y}) \exp(iyN)F \rightarrow \exp(iN)\hat{F}$ ($y \rightarrow \infty$). Since $\exp(iN)\hat{F}$ is a Hodge structure by [2, Lemma 3.12], $\nu(\sqrt{y}) \exp(iyN)F$ is a Hodge structure for any sufficiently large $y > 0$. This implies that $\exp(iyN)F$ is a Hodge structure for any sufficiently large $y > 0$.

Next we review the definition of the goodness based on [10, Section 7]. Here we adopt a simplified description. Later in 9.2 and 9.3, we discuss the equivalence of the definition here and that in [10, Section 7]. Let $f : Y \rightarrow 0$ be a proper, separated and log smooth fs log analytic space over 0. Assume that for any $y \in Y$, the cokernel of $\mathbf{Z} = M_0^{\text{gp}}/\mathcal{O}_0^{\times} \rightarrow M_{Y,y}^{\text{gp}}/\mathcal{O}_{Y,y}^{\times}$ is torsion-free.

DEFINITION 2.3. Let $q \geq 0$. We say that $Y/0$ is *good* for $\mathcal{H}^q$ if the following conditions (i)–(iii) are satisfied. We say that $Y/0$ is *good* if it is good for $\mathcal{H}^q$ for any q .

(i) The three homomorphisms

$$\begin{aligned}
\mathcal{O}_0^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} &\leftarrow \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} \tau_* (\mathcal{O}_0^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z}) \\
&\rightarrow \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} \tau_* R^q f_*^{\log} \omega_{Y/0}^{\log} \leftarrow \mathcal{O}_0^{\log} \otimes_{\mathbf{C}} R^q f_* \omega_{Y/0},
\end{aligned}$$

where the first homomorphism is induced by the adjoint map, the second one is by the natural map $f^{\log-1} \mathcal{O}_0^{\log} \rightarrow \omega_{Y/0}^{\log}$, and the third one is induced by the composite

$$\tau_* R^q f_*^{\log} \omega_{Y/0}^{\log} \leftarrow R^q f_* R \tau_* \omega_{Y/0}^{\log} \xrightarrow{\sim} R^q f_* \omega_{Y/0},$$

are isomorphisms.

Here, $\omega_{Y/0}$ is the relative analytic log de Rham complex.

(ii) For any integer r , the natural map $R^q f_* \omega_{Y/0}^{\geq r} \rightarrow R^q f_* \omega_{Y/0}$ is injective, and the natural map $R^q f_* \omega_{Y/0}^{\geq r} \rightarrow R^q f_* \omega_{Y/0}^r$ is surjective.

(iii) The isomorphisms in (i) together with the filtration induced by the injections in (ii) give a log Hodge structure of weight q .

2.4. We discuss the relationship with [10, Section 7].

First, note that there is a mistake in [10, 7.1]: The first arrow in the last line of *ibid.* p. 164 should be converted. Hence, the condition (i) there should be the same form of the above condition 2.3 (i), i.e., “The three homomorphisms...”

Assume that we made this modification on [10, 7.1]. Then, our definition of goodness 2.3 is precisely equivalent to the goodness of [10, 7.1], and also equivalent to another goodness of [10, 7.2] (note that our f is always exact because the log of the base 0 is just $\mathbf{N}$). We prove these equivalences in the last section.

More strongly, for $Y \rightarrow 0$ as above, the condition 2.3 (i) always holds, which we will prove in 6.1. Similarly, both the condition [10, 7.1 (i)] and the modified (i) in [10, 7.2] also always hold. Further, these two isomorphisms in the derived category coincide, as we will see in the last section (9.3).

2.5. We summarize the current situation of the problem of the goodness. See [10, Section 9] for details. Let $f : P \rightarrow S$ be a proper, separated, vertical and log smooth morphism of fs log analytic spaces. Assume that for any $p \in P$, the cokernel of $M_{S, f(p)}^{\text{gp}} / \mathcal{O}_{S, f(p)}^{\times} \rightarrow M_{P, p}^{\text{gp}} / \mathcal{O}_{P, p}^{\times}$ is torsion-free.

Then, we can expect that, under some Kähler type conditions, f is good either if S is log smooth or if f is exact. For the former case, there is a satisfactory result by Kato–Matsubara–Nakayama ([13]). On the other hand, there have not been known any nontrivial example if S is not log smooth. The main theorem 2.7 in this paper provides such examples for the first time.

2.6. A morphism $f : Y \rightarrow 0$ of fs log analytic spaces is called a *log deformation* ([21]) if Y is locally isomorphic to an open subspace of

$$\mathrm{Spec}(\mathbf{C}[t_1, \dots, t_{d+1}]/(t_1 \cdots t_{d+1})) \quad (d \geq 0)$$

endowed with the log structure defined by $\mathbf{N}^{d+1}$ sending e_j to t_j and endowed with the structure morphism sending $1 \in \mathbf{N}$ to $t_1 \cdots t_{d+1}$. Here $(e_j)_{1 \leq j \leq d+1}$ is the canonical basis of $\mathbf{N}^{d+1}$. (Equivalently, f is a log deformation if and only if Y is locally isomorphic to the log special fiber of a semistable family over a log pointed disk endowed with the natural log structure.)

A log deformation is said to be projective if the underlying space of Y is projective.

THEOREM 2.7. *A projective log deformation $f : Y \rightarrow 0$ is good.*

3. Review of [5]

In this section, we briefly review [5].

3.1. Let $f : Y \rightarrow 0$ be a projective log deformation. We describe the irreducible decomposition of Y by $Y = \bigcup_{\lambda \in \mathcal{A}} Y_\lambda$. We set

$$Y_{\underline{\lambda}} = Y_{\lambda_0} \cap Y_{\lambda_1} \cap \cdots \cap Y_{\lambda_k}$$

for a subset $\underline{\lambda} = \{\lambda_0, \lambda_1, \dots, \lambda_k\} \subset \mathcal{A}$ (for $\underline{\lambda} = \emptyset$, we set $Y_{\emptyset} = Y$) as in [5, 3.2]. The inclusion $Y_{\underline{\lambda}} \rightarrow Y$ is denoted by $a_{\underline{\lambda}}$. In what follows, we often omit the symbol $(a_{\underline{\lambda}})_*$ for short. A log structure on $Y_{\underline{\lambda}}$ is induced by $a_{\underline{\lambda}} : Y_{\underline{\lambda}} \rightarrow Y$. Thus an augmented cubical log analytic space $a : Y_\bullet \rightarrow Y$ is obtained. The relative log de Rham complex $\omega_{Y_{\underline{\lambda}}/0}$ for $\underline{\lambda} \subset \mathcal{A}$ defines a co-cubical complex $\omega_{Y_\bullet/0}$ on Y . On $Y_{\underline{\lambda}}$, a morphism of complexes

$$d \log t \wedge : \omega_{Y_{\underline{\lambda}}} \rightarrow \omega_{Y_{\underline{\lambda}}}[1]$$

is given by

$$\omega_{Y_{\underline{\lambda}}}^p \ni x \mapsto d \log t \wedge x \in \omega_{Y_{\underline{\lambda}}}^{p+1}$$

for every p . Then a complex $\mathbf{C}[u] \otimes \omega_{Y_{\underline{\lambda}}}$, where u is an indeterminate, is obtained by the differential

$$\mathrm{id} \otimes d + (2\pi\sqrt{-1})^{-1} \frac{d}{du} \otimes d \log t \wedge : \mathbf{C}[u] \otimes \omega_{Y_{\underline{\lambda}}}^p \rightarrow \mathbf{C}[u] \otimes \omega_{Y_{\underline{\lambda}}}^{p+1}$$

for every p . We denote it by $\omega_{Y_{\underline{\lambda}}}[u]$ for short. In particular, we have a complex $\omega_Y[u]$ on Y as the case of $\underline{\lambda} = \emptyset$. Thus we obtain a co-cubical complex $\omega_{Y_\bullet}[u]$ on Y .

3.2. As in [5, 5.5], we set

$$K_{\mathbf{C}} = \mathcal{C}(\omega_{Y_{\bullet}}[u]),$$

where $\mathcal{C}$ denotes the Čech complex of the co-cubical complex (see e.g. [5, 2.5]). We define the complex $K_{\mathbf{Q}}$ similarly by replacing $\omega_{Y_{\bullet}}$ with the Koszul complex on $Y_{\bullet}$. The increasing filtrations W on $K_{\mathbf{Q}}$ and $K_{\mathbf{C}}$, and the decreasing filtration F on $K_{\mathbf{C}}$ are defined in [5, 5.5]. Moreover the morphism $K_{\mathbf{Q}} \rightarrow K_{\mathbf{C}}$ is also defined in [5, 5.5]. Theorem 5.9 of [5] states that these data yield a mixed Hodge structure

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F)$$

for every q .

3.3. The morphism

$$\frac{d}{du} \otimes \text{id} : \mathbf{C}[u] \otimes \omega_{Y_2}^p \rightarrow \mathbf{C}[u] \otimes \omega_{Y_2}^p$$

induces an endomorphism of the co-cubical complex $\omega_{Y_{\bullet}}[u]$. Then a morphism of complexes

$$\frac{d}{du} \otimes \text{id} : K_{\mathbf{C}} \rightarrow K_{\mathbf{C}}$$

is induced. We obtain a morphism

$$H^q\left(Y, \frac{d}{du} \otimes \text{id}\right) : H^q(Y, K_{\mathbf{C}}) \rightarrow H^q(Y, K_{\mathbf{C}})$$

for every q , denoted by N_K in [5, 5.16]. Similarly, we obtain a morphism of complexes $K_{\mathbf{Q}} \rightarrow K_{\mathbf{Q}}$ and a morphism $H^q(Y, K_{\mathbf{Q}}) \rightarrow H^q(Y, K_{\mathbf{Q}})$ for every q . The latter is also denoted by N_K in [5, 5.16]. These morphisms are compatible with the morphism $K_{\mathbf{Q}} \rightarrow K_{\mathbf{C}}$ above. Theorem 8.16, the main result of [5], claims the following.

THEOREM 3.4. *For every q , there exists a bilinear form S_q on $H^q(Y, K_{\mathbf{Q}})$ such that*

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F, N_K, S_q)$$

is a polarized mixed Hodge structure in the sense of Cattani–Kaplan–Schmid [2, Definition (2.26)].

4. $\mathbf{Q}$ -structures

The main step of our proof of Theorem 2.7 is to show the compatibility of the two $\mathbf{Q}$ -structures: that in 2.3 and that by [5] reviewed in Section 3.

To prove this compatibility, we introduce an intermediate $\mathbf{Q}$ -structure in this section, and later, in Sections 5 and 7, we prove that this intermediate $\mathbf{Q}$ -structure coincides with each $\mathbf{Q}$ -structure of two, respectively.

4.1. Fix an integer q . Let $H_{\mathbf{Q}} = R^q f_*^{\log} \mathbf{Q}$. Let $H_{\mathcal{O}} = H^q(Y, \omega_{Y/0}^{\log})$. We will prove in 6.1 that we have the quasi-isomorphisms

$$\mathcal{O}_0^{\log} \otimes H_{\mathbf{Q}} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes \tau_*(\mathcal{O}_0^{\log} \otimes H_{\mathbf{Q}}) \xrightarrow{\sim} \mathcal{O}_0^{\log} \otimes \tau_* R^q f_*^{\log} \omega_{Y/0}^{\log} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes H_{\mathcal{O}} \quad (1)$$

as in 2.3 (i).

Let z be the point of $0^{\log}$ corresponding to 1. Taking the stalk at z of (1) and specializing it by the ring homomorphism $\mathcal{O}_{0,z}^{\log} = \mathbf{C} \left[\frac{\log t}{2\pi i} \right] \rightarrow \mathbf{C}; \frac{\log t}{2\pi i} \mapsto 0$, we have an isomorphism

$$\mathbf{C} \otimes H_{\mathbf{Q},z} \cong H_{\mathcal{O}}. \quad (2)$$

Then, the main part of the proof of Theorem 2.7 is the following statement.

4.1.1. The $\mathbf{Q}$ -structure defined by (2) coincides with the $\mathbf{Q}$ -structure defined by the isomorphism

$$\mathbf{C} \otimes H^q(Y, K_{\mathbf{Q}}) \cong H^q(Y, K_{\mathbf{C}}) \cong H_{\mathcal{O}} \quad (3)$$

constructed in [5, Theorem 5.9].

4.2. To prove 4.1.1, we introduce an intermediate $\mathbf{Q}$ -structure.

Consider the following commutative diagram of ringed spaces:

$$\begin{array}{ccccccc} Y_z^{\log} & \xrightarrow{i} & Y_{\infty}^{\log} & \xrightarrow{\pi} & Y^{\log} & \xrightarrow{\tau} & Y \\ \downarrow f^{\log} & & \downarrow f_{\infty} & & \downarrow f^{\log} & & \downarrow f \\ \{0\} & \xrightarrow{i} & \mathbf{R} & \xrightarrow{\pi} & 0^{\log} & \xrightarrow{\tau} & 0, \end{array}$$

where the bottom π is the map $\mathbf{R} \ni r \mapsto e^{2\pi i r} \in \{\alpha \mid \alpha \bar{\alpha} = 1\} = 0^{\log}$, and the left two squares are cartesian.

Consider the composite of quasi-isomorphisms

$$R(\tau\pi)_* \mathbf{C} \xrightarrow{\sim} R(\tau\pi)_* \pi^{-1} \omega_Y^{\log} \xleftarrow{\sim} \omega_Y[u] \xrightarrow{\sim} \omega_{Y/0}^{\log} \quad (4)$$

on Y , where the first quasi-isomorphism is by log Poincaré lemma ([14, Theorem (3.8)]), the second is by [6, Lemma 3.3], and the last one is by [14, Remark (4.10)] and sends the indeterminate u to 0.

Consider its $R^q f_*$ on 0:

$$\mathbf{C} \otimes H^q(Y_{\infty}^{\log}, \mathbf{Q}) = H^q(Y_{\infty}^{\log}, \mathbf{C}) \cong H_{\mathcal{O}}. \quad (5)$$

Then, the statement 4.1.1 reduces to the following two statements:

4.2.1. The $\mathbf{Q}$ -structure defined by (3) coincides with the $\mathbf{Q}$ -structure defined by (5).

4.2.2. The $\mathbf{Q}$ -structure defined by (2) coincides with the $\mathbf{Q}$ -structure defined by (5).

We will prove 4.2.1 in Section 5, and 4.2.2 in Section 7.

5. Double complexes

5.1. In this section, we prove 4.2.1. The strategy is as follows. Just as Steenbrink uses the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{Q}\text{-nearby cycle} & \longrightarrow & A_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ \mathbf{C}\text{-nearby cycle} & \longrightarrow & \Omega_{X/S}(\log Y) \otimes \mathcal{O}_Y \longrightarrow A_{\mathbf{C}} \end{array}$$

in the proof of [20, Theorem (4.19)] (notation be as there), we construct the commutative diagram

$$\begin{array}{ccc} R(\tau\pi)_* \mathbf{Q} & \longrightarrow & K_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ R(\tau\pi)_* \mathbf{C} & \xrightarrow{(4)} \omega_{Y/0} \longrightarrow & K_{\mathbf{C}}. \end{array}$$

Then, by taking $R^q f_*$, we have 4.2.1.

To this end, we recall several definitions in [6] and correct small mistakes on the sign of the differentials.

DEFINITION 5.2. Let (K_n, W) be a family of complexes equipped with an increasing filtration W for every non-negative integer n , and $\theta_n : K_n \rightarrow K_{n+1}[1]$, $n \geq 0$, a morphism of complexes, such that the conditions (1.4.1)–(1.4.5) in [6, Definition 1.4] are satisfied. Then we set

- $D^{p,q} = K_q^{p+q+1}/W_q$ for $p, q \in \mathbb{Z}_{\geq 0}$
- $d' : D^{p,q} \rightarrow D^{p+1,q}$ is the morphism induced by $-d_{K_q}$, where d_{K_q} denotes the differential of the complex K_q
- $d'' : D^{p,q} \rightarrow D^{p,q+1}$ is induced by the morphism $-\theta_q : K_q \rightarrow K_{q+1}[1]$

for every $p, q \in \mathbb{Z}_{\geq 0}$. Then we can easily see the equality $d'd'' + d''d' = 0$. Thus we obtain a double complex $D = D(K_n, W, \theta_n)$. The associated single complex is denoted by $sD(K_n, W, \theta_n)$ and called the Steenbrink complex of $\{(K_n, W), \theta_n\}$. The Steenbrink complex $sD(K_n, W, \theta_n)$ is equipped with the increasing filtration L defined in [6, Definition 1.4] and with the decreasing filtration F defined in [6, Remark 1.6].

For the case where there exist K and $\theta : K \rightarrow K[1]$ such that $(K_n, W) = (K, W)$ and $\theta_n = \theta$ for all n , $sD(K_n, W, \theta_n)$ is simply denoted by $sD(K, W, \theta)$ in this article.

REMARK 5.3. Compared with Definition 1.4 in [6], we change the signs of d' and d'' . By the definition above, we obtain the formula

$$\mathrm{gr}_m^L sD(K_n, W, \theta_n) = \bigoplus_{\substack{q \geq 0 \\ q \geq -m}} \mathrm{gr}_{m+2q+1}^W K_q[1]$$

for every m .

REMARK 5.4. As described in [6, Remark 1.7], the construction of the Steenbrink complex satisfies the natural functoriality. In particular, we have the following functoriality:

Let $\{(K_n, W), \theta_n\}$ be as above, K' a complex, and $f : K' \rightarrow K_0[1]$ a morphism of complexes satisfying the condition $\theta_0 f = 0$. Then the morphism

$$(K')^p \rightarrow D(K_n, W, \theta_n)^{p,0} = K_0^{p+1}/W_0$$

induced by f defines a morphism of complexes $K' \rightarrow sD(K_n, W, \theta_n)$.

DEFINITION 5.5. Let $\delta : K \rightarrow K(-1)$ be a morphism of complexes. A complex $\rho(K, \delta)$ is defined by $\rho(K, \delta)^p = K^p \oplus K^{p-1}(-1)$ with the differential $d : \rho(K, \delta)^p \rightarrow \rho(K, \delta)^{p+1}$ defined by $d(x, y) = (dx, -dy + \delta(x))$ for $(x, y) \in K^p \oplus K^{p-1}(-1) = \rho(K, \delta)^p$. The morphisms of complexes $\theta : \rho(K, \delta) \rightarrow \rho(K, \delta)(1)[1]$ and $\mu : K \rightarrow \rho(K, \delta)(1)[1]$ are defined by

$$\theta(x, y) = (0, x) \in K^{p+1}(1) \oplus K^p = \rho(K, \delta)^{p+1}(1)$$

for $(x, y) \in K^p \oplus K^{p-1}(-1) = \rho(K, \delta)^p$ and

$$\mu(x) = (0, x) \in K^{p+1}(1) \oplus K^p = \rho(K, \delta)^{p+1}(1)$$

for $x \in K^p$. It is easy to check that they actually define morphisms of complexes.

For $K_n = \rho(K, \delta)(n+1)$ with the canonical filtration $W = \tau$ and $\theta_n = \theta(n+1)$, the Steenbrink complex $sD(K_n, W, \theta_n)$ is denoted by $SZ(K, \delta)$ and called the Steenbrink–Zucker complex for (K, δ) . Then we obtain a morphism of complexes

$$\mu : K \rightarrow SZ(K, \delta) \tag{6}$$

by Remark 5.4 because we have $\theta_0 \mu = \theta(1) \mu = 0$ by definition.

REMARK 5.6. The definition of the morphism μ is different from that in [6, Remark 1.13] because the sign of differential of the Steenbrink complex in Definition 5.2 is changed.

5.7. Let $f : Y \rightarrow 0$ be a log deformation. We fix an injective resolution $\mathbf{Q}_{Y^{\log}} \rightarrow I$ of $\mathbf{Q}_{Y^{\log}}$. Then we have the monodromy automorphism $T : (\tau\pi)_*\pi^{-1}I \rightarrow (\tau\pi)_*\pi^{-1}I$. A subcomplex $B(I)$ of $(\tau\pi)_*\pi^{-1}I$ is defined by

$$B(I) = \bigcup_{m \geq 0} (\text{Ker}(T - \text{id})^{m+1}) \subset (\tau\pi)_*\pi^{-1}I$$

in [6, (3.4.1)]. The inclusion $B(I) \rightarrow (\tau\pi)_*\pi^{-1}I$ is a quasi-isomorphism by [6, Lemma 3.5]. Moreover, $\log T$ is well-defined on $B(I)$ by definition.

5.8. We consider the commutative diagram

$$\begin{array}{ccc} L^0 & \xrightarrow{\varepsilon} & L^1 \\ v_0 \downarrow & & \downarrow v_1 \\ \mathcal{O}_Y & \xrightarrow{\mathbf{e}} & M_Y^{\text{gp}} \end{array} \quad (7)$$

of abelian sheaves on Y which appeared in [6, (5.1.1)], where L^0 and L^1 are torsion-free abelian sheaves and the morphism $\mathbf{e} : \mathcal{O}_Y \rightarrow M_Y^{\text{gp}}$ is defined by $\mathbf{e}(f) = \exp(2\pi\sqrt{-1}f)$ for $f \in \mathcal{O}_Y$. As in [6, (5.1.2)], we assume that there exists an element of $\Gamma(Y, L^1)$ whose image by v_1 is $t \in \Gamma(Y, M_Y)$. We denote this element by $t \in \Gamma(Y, L^1)$ by abuse of the notation.

We obtain the Koszul complexes $\text{Kos}^n(\varepsilon)$ and $\text{Kos}^n(\mathbf{e})$ by

$$\text{Kos}^n(\varepsilon)^p = \Gamma_{n-p}(L^0) \otimes \bigwedge^p L^1$$

$$\text{Kos}^n(\mathbf{e})^p = \Gamma_{n-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\text{gp}}$$

as in [21, Section 1]. (For the details, see [21], [5], [6]). An increasing filtration W on $\text{Kos}^n(\varepsilon)$ is defined in [21, (2.8)]. Similarly, an increasing filtration W is defined on $\text{Kos}^n(\mathbf{e})$ (see also [5, 3.15]). Moreover, the morphism

$$\Gamma_{n-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\text{gp}} \ni x \otimes y \mapsto 1^{[1]}x \otimes y \in \Gamma_{n+1-p}(\mathcal{O}_Y) \otimes \bigwedge^p M_Y^{\text{gp}}$$

defines a morphism of complexes $\text{Kos}^n(\mathbf{e}) \rightarrow \text{Kos}^{n+1}(\mathbf{e})$ as [5, (3.11.2)]. Thus we obtain an inductive system

$$\cdots \rightarrow \text{Kos}^n(\mathbf{e}) \rightarrow \text{Kos}^{n+1}(\mathbf{e}) \rightarrow \cdots$$

and set

$$\text{Kos}_Y(M_Y) = \varinjlim_n \text{Kos}^n(\mathbf{e})$$

as in [5, 3.11]. The increasing filtration W is obtained by

$$W_m \operatorname{Kos}_Y(M_Y) = \varinjlim_n W_m \operatorname{Kos}^n(\mathbf{e})$$

for every m .

The morphism

$$\Gamma_{n-p}(L^0) \otimes \bigwedge^p L^1 \ni x \otimes y \mapsto x \otimes t \wedge y \in \Gamma_{n-p}(L^0) \otimes \bigwedge^{p+1} L^1$$

defines a morphism of complexes

$$\operatorname{Kos}^n(\varepsilon) \rightarrow \operatorname{Kos}^{n+1}(\varepsilon)[1]$$

denoted by $t \wedge$ for short. Similarly, we obtain a morphism of complexes

$$t \wedge : \operatorname{Kos}^n(\mathbf{e}) \rightarrow \operatorname{Kos}^{n+1}(\mathbf{e})[1].$$

Moreover, the morphism

$$t \wedge : \operatorname{Kos}_Y(M_Y) \rightarrow \operatorname{Kos}_Y(M_Y)$$

is similarly defined in [5, (3.13.1)].

From the commutative diagram (7), the morphism of complexes

$$\operatorname{Kos}^n(\varepsilon) \rightarrow \operatorname{Kos}^n(\mathbf{e})$$

is obtained for every n . On the other hand, we have the canonical morphism

$$\operatorname{Kos}^n(\mathbf{e}) \rightarrow \operatorname{Kos}_Y(M_Y) = \varinjlim_n \operatorname{Kos}^n(\mathbf{e})$$

for every n . These morphisms preserve the filtrations W by definition. The morphism of complexes $\psi : \operatorname{Kos}_Y(M_Y) \rightarrow \omega_Y$ is defined as in [5, (3.11.3)], which preserves the filtration W on the both sides. Then we have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Kos}^n(\varepsilon) & \xrightarrow{t \wedge} & \operatorname{Kos}^{n+1}(\varepsilon)[1] \\ \downarrow & & \downarrow \\ \operatorname{Kos}^n(\mathbf{e}) & \xrightarrow{t \wedge} & \operatorname{Kos}^{n+1}(\mathbf{e})[1] \\ \downarrow & & \downarrow \\ \operatorname{Kos}_Y(M_Y) & \xrightarrow{t \wedge} & \operatorname{Kos}_Y(M_Y)[1] \\ \downarrow \psi & & \downarrow \psi \\ \omega_Y & \xrightarrow{(2\pi\sqrt{-1})^{-1} d \log t \wedge} & \omega_Y[1], \end{array}$$

where the commutativity of the bottom square is proved in [5, (3.13.2)]. The composite $\text{Kos}^n(\varepsilon) \rightarrow \omega_Y$ of the vertical arrows coincides with the morphism ϕ defined in [6, (5.1.4)].

DEFINITION 5.9. In the situation above, we set

$$A_{\mathbf{Q}} = sD(\text{Kos}_Y(M_Y), W, t\wedge), \quad A_{\mathbf{C}} = sD(\omega_Y, W, d \log t\wedge)$$

as in [5, (5.17)].

On the other hand, we set

$$K_n = \text{Kos}^{n_0+n+1}(\varepsilon)(n+1), \quad \theta_n = (2\pi\sqrt{-1})t\wedge : K_n \rightarrow K_{n+1}[1],$$

where $n_0 = \dim Y + 1$. Thus we obtain the Steenbrink complex $sD(K_n, W, \theta_n)$.

We have the following commutative diagram:

$$\begin{array}{ccc} \text{Kos}^{n_0+n+1}(\varepsilon)(n+1) & \xrightarrow{(2\pi\sqrt{-1})t\wedge} & \text{Kos}^{n_0+n+2}(\varepsilon)(n+2)[1] \\ \downarrow (2\pi\sqrt{-1})^{-n-1} \text{id} & & \downarrow (2\pi\sqrt{-1})^{-n-2} \text{id} \\ \text{Kos}^{n_0+n+1}(\varepsilon) & \xrightarrow{t\wedge} & \text{Kos}^{n_0+n+2}(\varepsilon)[1] \\ \downarrow & & \downarrow \\ \text{Kos}_Y(M_Y) & \xrightarrow{t\wedge} & \text{Kos}_Y(M_Y)[1] \\ \downarrow (2\pi\sqrt{-1})^{n+1} \psi & & \downarrow (2\pi\sqrt{-1})^{n+2} \psi \\ \omega_Y & \xrightarrow{d \log t\wedge} & \omega_Y[1], \end{array}$$

where the both composites of the vertical arrows coincide with ϕ . Therefore we obtain the commutative diagram

$$\begin{array}{ccc} sD(K_n, W, \theta_n) & \longrightarrow & A_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} \end{array} \tag{8}$$

of the Steenbrink complexes. The vertical arrows $sD(K_n, W, \theta_n) \rightarrow A_{\mathbf{C}}$ and $A_{\mathbf{Q}} \rightarrow A_{\mathbf{C}}$ are denoted by $sD(\phi)$ and α respectively.

REMARK 5.10. The triple $((A_{\mathbf{Q}}, L), (A_{\mathbf{C}}, L, F), \alpha)$ above coincides with (A, W, F) defined in [5, Definition 5.18]. On the other hand, the triple

$$((sD(K_n, W, \theta_n), L), (A_{\mathbf{C}}, L, F), sD(\phi))$$

coincides with (5.1.5) in [6] except the sign of the differentials.

5.11. As in [6, (3.4)] and the proof of [6, Lemma (3.5)], we have the commutative diagram

$$\begin{array}{ccccccc} R(\tau\pi)_* \mathbf{Q} & \xrightarrow{\sim} & (\tau\pi)_* \pi^{-1} I & \xleftarrow{\sim} & B(I) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ R(\tau\pi)_* \mathbf{C} & \xrightarrow{\sim} & (\tau\pi)_* \pi^{-1} I_{\mathbf{C}} & \xleftarrow{\sim} & B(I)_{\mathbf{C}} & \xleftarrow{\sim} & \omega_Y[u] \end{array}$$

whose horizontal arrows are quasi-isomorphisms. The diagram

$$\begin{array}{ccc} \omega_Y[u] & \xrightarrow{\frac{d}{du}} & \omega_Y[u] \\ \downarrow & & \downarrow \\ B(I)_{\mathbf{C}} & \xrightarrow{\log T} & B(I)_{\mathbf{C}} \end{array} \quad (9)$$

is commutative as stated in [6, (3.6)]. We set $\delta = -(2\pi\sqrt{-1})^{-1} \log T$ on $B(I)$ and $\delta = -(2\pi\sqrt{-1})^{-1} \frac{d}{du}$ on $\omega_Y[u]$ for the simplicity of notation.

The morphism of complexes

$$SZ(B(I), \delta) \rightarrow SZ(B(I)_{\mathbf{C}}, \delta) = SZ(B(I), \delta)_{\mathbf{C}}$$

coincides with the natural morphism induced by the base extension $\mathbf{Q} \hookrightarrow \mathbf{C}$. Moreover, we obtain the morphism of complexes

$$SZ(\omega_Y[u], \delta) \rightarrow SZ(B(I)_{\mathbf{C}}, \delta)$$

by the commutative diagram (9). It turns out that this is a (filtered) quasi-isomorphism by [6, Lemma 3.7].

On the other hand, the morphism

$$\varphi : \rho(\omega_Y[u], \delta)^p \rightarrow \omega_Y^p$$

is defined by

$$\varphi(x, y) = x_0 + d \log t \wedge y_0$$

for $(x, y) \in \omega_Y^p[u] \oplus \omega_Y^{p-1}[u]$, where $x = \sum_j x_j (u^j/j!)$ and $y = \sum_j y_j (u^j/j!)$. It is easy to check that this defines a morphism of complexes $\varphi : \rho(\omega_Y[u], \delta) \rightarrow \omega_Y$. (This morphism is denoted by ψ in [6, (3.9.1)].) Moreover, we can easily check that the diagram

$$\begin{array}{ccc} \rho(\omega_Y[u], \delta) & \xrightarrow{\theta} & \rho(\omega_Y[u], \delta)[1] \\ \downarrow \varphi & & \downarrow \varphi[1] \\ \omega_Y & \xrightarrow{d \log t \wedge} & \omega_Y[1] \end{array}$$

is commutative. Thus the morphism of complexes

$$SZ(\omega_Y[u], \delta) = sD(\rho(\omega_Y[u], \delta), \tau, \theta) \rightarrow sD(\omega_Y, W, d \log t \wedge) = A_{\mathbf{C}}$$

is obtained by the functoriality of the Steenbrink complexes. The diagram

$$SZ(B(I)_{\mathbf{C}}, \delta) \xleftarrow{\sim} SZ(\omega_Y[u], \delta) \rightarrow A_{\mathbf{C}} \quad (10)$$

defines a morphism

$$\alpha : SZ(B(I)_{\mathbf{C}}, \delta) \rightarrow A_{\mathbf{C}}$$

in the (filtered) derived category.

THEOREM 5.12. *The data*

$$((A_{\mathbf{Q}}, L), (A_{\mathbf{C}}, L, F), \alpha), \quad (11)$$

$$((SZ(B(I), \delta), L), (A_{\mathbf{C}}, L, F), \alpha), \quad (12)$$

$$(sD(K_n, W, \theta_n), L), (A_{\mathbf{C}}, L, F), sD(\phi)) \quad (13)$$

are cohomological mixed Hodge complexes on Y .

PROOF. For $((A_{\mathbf{Q}}, L), (A_{\mathbf{C}}, L, F), \alpha)$, Theorem 5.21 of [5] implies the conclusion. For the other two, the same proofs as those for Theorem 3.12 in [6] and for (5.4) in [21] work.

5.13. In order to relate (12) and (13), the commutative diagram in the derived category

$$\begin{array}{ccccc} SZ(B(I), \delta) & \xleftarrow{\sim} & SZ(\text{Kos}^{n_0}(\tilde{e}), \delta) & \xrightarrow{\sim} & sD(K_n, W, \theta_n) \\ \downarrow & & \downarrow & & \downarrow \\ A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} \end{array} \quad (14)$$

is constructed in the paragraph before Proposition 5.5 and in the proof of Proposition 5.7 in [6], where $((SZ(\text{Kos}^{n_0}(\tilde{e}), \delta), L), (A_{\mathbf{C}}, L, F))$ together with the middle vertical morphism is a cohomological mixed Hodge complex in [6, Proposition 5.5].

5.14. The comparison morphisms

$$A_{\mathbf{Q}} \rightarrow K_{\mathbf{Q}}, \quad A_{\mathbf{C}} \rightarrow K_{\mathbf{C}} \quad (15)$$

defined in Definition 5.26 and Definition 5.24 in [5] give us the commutative diagram

$$\begin{array}{ccc}
 A_{\mathbf{Q}} & \longrightarrow & K_{\mathbf{Q}} \\
 \downarrow & & \downarrow \\
 A_{\mathbf{C}} & \longrightarrow & K_{\mathbf{C}}
 \end{array} \tag{16}$$

by [5, Lemma 5.27].

Combining the commutative diagrams (8), (14) and (16), we obtain the commutative diagram

$$\begin{array}{ccccccccc}
 SZ(B(I), \delta) & \longleftarrow & SZ(\mathrm{Kos}^{n_0}(\tilde{\mathcal{E}}), \delta) & \longrightarrow & sD(K_n, W, \theta_n) & \longrightarrow & A_{\mathbf{Q}} & \longrightarrow & K_{\mathbf{Q}} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} & \xlongequal{\quad} & A_{\mathbf{C}} & \longrightarrow & K_{\mathbf{C}}
 \end{array}$$

in the derived category.

5.15. Now it suffices to construct a commutative diagram

$$\begin{array}{ccc}
 R(\tau\pi)_* \mathbf{Q} & \longrightarrow & SZ(B(I), \delta) \\
 \downarrow & & \downarrow \alpha \\
 R(\tau\pi)_* \mathbf{C} & \xrightarrow{(4)} \omega_{Y/0} \longrightarrow & A_{\mathbf{C}}
 \end{array}$$

in the derived category.

Because we have an isomorphism

$$B(I) \rightarrow R(\tau\pi)_* \mathbf{Q}$$

in the derived category, the diagram

$$R(\tau\pi)_* \mathbf{Q} \xleftarrow{\sim} B(I) \xrightarrow{\mu} SZ(B(I), \delta) \tag{17}$$

defines a morphism $R(\tau\pi)_* \mathbf{Q} \rightarrow SZ(B(I), \delta)$ in the derived category, where μ denotes the morphism defined in (6).

On the other hand, the morphism

$$d \log t \wedge : \omega_Y \rightarrow \omega_Y[1]$$

induces a morphism $\omega_Y \rightarrow A_{\mathbf{C}}$ as in Remark 5.4. Then we obtain a morphism

$$\omega_{Y/0} \rightarrow A_{\mathbf{C}} \tag{18}$$

because the morphism $\omega_Y \rightarrow A_{\mathbf{C}}$ clearly factors through the morphism $\omega_Y \rightarrow \omega_{Y/0}$.

What we have to prove is the following:

PROPOSITION 5.16. *The morphisms (17) and (18) make the diagram (5.15) commutative.*

PROOF. It is sufficient to prove that the diagram

$$\begin{array}{ccccc}
 R(\tau\pi)_* \mathbf{Q} & \xleftarrow{\sim} & B(I) & \xrightarrow{\mu} & SZ(B(I), \delta) \\
 \downarrow & & \downarrow & & \downarrow \\
 R(\tau\pi)_* \mathbf{C} & \xleftarrow{\sim} & B(I)_{\mathbf{C}} & \xrightarrow{\mu} & SZ(B(I), \delta)_{\mathbf{C}} \\
 & & \uparrow \wr & & \uparrow \wr \\
 & & \omega_Y[u] & \xrightarrow{\mu} & SZ(\omega_Y[u], \delta) \\
 & & \downarrow & & \downarrow sD(\varphi) \\
 & & \omega_{Y/0} & \longrightarrow & A_{\mathbf{C}}
 \end{array} \tag{19}$$

is commutative because the morphism (4) is represented by

$$R(\tau\pi)_* \mathbf{C} \simeq (\tau\pi)_* \pi^{-1} I_{\mathbf{C}} \xleftarrow{\sim} B(I)_{\mathbf{C}} \xleftarrow{\sim} \omega_Y[u] \rightarrow \omega_{Y/0}$$

in the derived category and because the morphism $\alpha: SZ(B(I), \delta) \rightarrow A_{\mathbf{C}}$ is represented as the composite of the right vertical arrows in (19) in the derived category. The squares except the bottom are trivially commutative by definition.

The diagram

$$\begin{array}{ccc}
 \omega_Y^p[u] & \xrightarrow{\mu} & \omega_Y^{p+1}[u] \oplus \omega_Y^p[u] \\
 \downarrow & & \downarrow \varphi \\
 \omega_Y^p & \xrightarrow{d \log t \wedge} & \omega_Y^{p+1}
 \end{array}$$

is commutative because we have

$$\varphi(\mu(x)) = \varphi(0, x) = d \log t \wedge x_0$$

for $x = \sum x_j(u^j/j!) \in \omega_Y^p[u]$. Then we can easily check the commutativity of the bottom square in the diagram (19).

5.17. Now we prove the coincidence of the morphisms N_K on $H^q(Y, K_{\mathbf{Q}})$ and the logarithm of the monodromy automorphism on $H_{\mathbf{Q}, z}$ under the identification above.

It is known that the monodromy automorphism on the stalk $H_{\mathbf{Q}, z}$ is induced from the monodromy automorphism on $\Gamma(\mathbf{R}, \pi^{-1} H_{\mathbf{Q}}) \simeq (\tau\pi)_* \pi^{-1} H_{\mathbf{Q}}$

(see e.g. Iversen [8, IV Theorem 9.7]). Thus the monodromy automorphism is induced by

$$T : (\tau\pi)_* \pi^{-1} I \rightarrow (\tau\pi)_* \pi^{-1} I$$

because

$$\begin{aligned} (\tau\pi)_* \pi^{-1} H_{\mathbf{Q}} &\simeq (\tau\pi)_* \pi^{-1} R^q f_*^{\log} \mathbf{Q} \simeq (\tau\pi)_* R^q (f_{\infty})_* \mathbf{Q} \\ &\simeq R^q (f \tau\pi)_* \mathbf{Q} \simeq R^q f_* (\tau\pi)_* \pi^{-1} I \simeq H^q(Y, B(I)) \end{aligned}$$

by the base change.

Therefore, it is sufficient to prove that $\log T$ on $H^q(Y, B(I)_{\mathbf{C}})$ coincides with N_K on $H^q(Y, K_{\mathbf{C}})$ under the identification induced by the morphism

$$B(I)_{\mathbf{C}} \xleftarrow{\sim} \omega_Y[u] \rightarrow \omega_{Y/0} \rightarrow A_{\mathbf{C}} \rightarrow K_{\mathbf{C}}$$

in the derived category. We already know that $\log T$ on $B(I)_{\mathbf{C}}$ is identified with $\frac{d}{du}$ on $\omega_Y[u]$ as in [6]. On the other hand, the canonical morphism $a : Y_{\bullet} \rightarrow Y$ induces the morphism

$$a^* : \omega_Y[u] \rightarrow K_{\mathbf{C}}$$

by definition. It is easy to see that the diagram

$$\begin{array}{ccc} \omega_Y[u] & \xrightarrow{\frac{d}{du}} & \omega_Y[u] \\ a^* \downarrow & & \downarrow a^* \\ K_{\mathbf{C}} & \xrightarrow{\frac{d}{du} \otimes \text{id}} & K_{\mathbf{C}} \end{array}$$

is commutative. Therefore the following lemma implies the conclusion.

LEMMA 5.18. *The morphism*

$$H^q(Y, a^*) : H^q(Y, \omega_Y[u]) \rightarrow H^q(Y, K_{\mathbf{C}})$$

coincides with the isomorphism induced by the morphisms

$$\omega_Y[u] \rightarrow \omega_{Y/0} \rightarrow A_{\mathbf{C}} \rightarrow K_{\mathbf{C}}$$

above for any q .

PROOF. The canonical morphism $a : Y_{\bullet} \rightarrow Y$ induces the morphism

$$a^* : \omega_{Y/0} \rightarrow \mathcal{C}(\omega_{Y_{\bullet}/0})$$

as in [5, 5.7], where $\mathcal{C}$ denotes the Čech complex of the co-cubical complex $\omega_{Y_{\bullet}/0}$ as in 3.2 (see e.g. [5, 2.5]). Similarly to the morphism $\omega_Y[u] \rightarrow \omega_{Y/0}$

in (4), we have a morphism of complexes

$$\omega_{Y_\lambda}[u] \rightarrow \omega_{Y_\lambda/0}$$

for every $\lambda \in \mathcal{A}$. They form a morphism of co-cubical complexes $\omega_{Y_\bullet}[u] \rightarrow \omega_{Y_\bullet/0}$. Thus the morphism of complexes

$$K_C \rightarrow \mathcal{C}(\omega_{Y_\bullet/0})$$

is obtained in [5, 5.5]. As in the proof of Theorem 5.29 in [5], we can easily check that the diagram

$$\begin{array}{ccccc} \omega_Y[u] & \longrightarrow & \omega_{Y/0} & \longrightarrow & A_C \\ a^* \downarrow & & a^* \downarrow & & \downarrow \\ K_C & \longrightarrow & \mathcal{C}(\omega_{Y_\bullet/0}) & \longleftarrow & K_C \end{array}$$

is commutative. Therefore we obtain the conclusion because the morphism $K_C \rightarrow \mathcal{C}(\omega_{Y_\bullet/0})$ above induces the isomorphism

$$H^q(Y, K_C) \xrightarrow{\sim} H^q(Y, \mathcal{C}(\omega_{Y_\bullet/0}))$$

for any q by Theorem 5.9 in [5].

6. Log Riemann–Hilbert correspondences

In this section, we prove that the condition of 2.3 (i) is in fact always satisfied. This is essentially a part of [7]. In the course of the proof, we also prove some facts which will be used to prove 4.2.2.

PROPOSITION 6.1. *Let $f : Y \rightarrow 0$ be a proper, separated and log smooth fs log analytic space over 0. Assume that for any $y \in Y$, the cokernel of $\mathbf{Z} = M_0^{\text{gp}}/\mathcal{O}_0^\times \rightarrow M_{Y,y}^{\text{gp}}/\mathcal{O}_{Y,y}^\times$ is torsion-free. Then, $Y \rightarrow 0$ satisfies the condition (i) in 2.3.*

For a later use, we prove the following generalized statement.

PROPOSITION 6.2. *Let $f : P \rightarrow S$ be a proper, separated, exact and log smooth morphism of fs log analytic spaces. Assume that for any $p \in P$, the cokernel of $M_{S,f(p)}^{\text{gp}}/\mathcal{O}_{S,f(p)}^\times \rightarrow M_{P,p}^{\text{gp}}/\mathcal{O}_{P,p}^\times$ is torsion-free. Let $q \geq 0$. Then the following hold.*

(i) *The three homomorphisms*

$$\begin{aligned} \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} &\leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z}) \\ &\rightarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_* R^q f_*^{\log} \omega_{P/S}^{\log} \leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S}, \end{aligned}$$

where the first homomorphism is induced by the adjoint map, the second one is by the natural map $f^{\log-1} \mathcal{O}_S^{\log} \rightarrow \omega_{P/S}^{\log}$ together with the projection formula, and the third one is induced by the composite

$$\tau_* R^q f_*^{\log} \omega_{P/S}^{\log} \leftarrow R^q f_* R \tau_* \omega_{P/S}^{\log} \xleftarrow{\sim} R^q f_* \omega_{P/S}^{\log},$$

are isomorphisms.

(ii) Two homomorphisms

$$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} \rightarrow R^q f_*^{\log} \omega_{P/S}^{\log} \leftarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S},$$

where the first homomorphism is induced by the projection formula and the natural map $f^{\log-1} \mathcal{O}_S^{\log} \rightarrow \omega_{P/S}^{\log}$ and the second one is by the projection formula, are isomorphisms. Further the composite of these two coincides with the composite of three in (i).

6.3. We prepare for the proof of the above proposition.

First we show the following vanishing results.

$$R^p \tau_* R^q f_*^{\log} \mathcal{O}_S^{\log} = 0 \quad (p > 0). \quad (20)$$

$$R^p \tau_* R^q f_*^{\log} \omega_{P/S}^{\log} = 0 \quad (p > 0). \quad (21)$$

As for (20), we note that, by [7, Theorem (6.3)], $R^q f_*^{\log} \mathcal{O}_S^{\log}$ is isomorphic to $\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} V^q$ for some $\mathcal{O}_S$ -module V^q . Then, [7, Proposition (3.7) (3)] implies the desired vanishing (20).

As for (21), we use the following proposition.

Recall that a morphism $f : P \rightarrow S$ is said to be log injective if for any $p \in P$, the homomorphism $M_{S,f(p)}/\mathcal{O}_{S,f(p)}^{\times} \rightarrow M_{P,p}/\mathcal{O}_{P,p}^{\times}$ is injective (cf. [18, Definition (5.5.1)]).

PROPOSITION 6.4. *Let $f : P \rightarrow S$ be a proper, separated and log injective morphism of fs log analytic spaces. Assume that for any $p \in P$, the cokernel of $M_{S,f(p)}/\mathcal{O}_{S,f(p)}^{\times} \rightarrow M_{P,p}^{\text{gp}}/\mathcal{O}_{P,p}^{\times}$ is torsion-free. Let F be an $\mathcal{O}_P$ -module. Then the natural homomorphism*

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* F \rightarrow Rf_*^{\log} (\mathcal{O}_P^{\log} \otimes_{\tau^{-1}\mathcal{O}_P} \tau^{-1} F)$$

is a quasi-isomorphism.

We remark that the case where f is exact and $F = \mathcal{O}_P$ of this proposition is explained in [10, 8.13]. The proof below is essentially the same.

PROOF. Factor f into $P \xrightarrow{\nu} P' \xrightarrow{f'} S$ such that the underlying morphism of ν is the identity and that f' is strict. Note that both ν and f' are proper,

separated and log injective. Then, we have

$$\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F \xrightarrow{\sim} R\nu_*^{\log}(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F), \quad (22)$$

i.e., 6.4 holds for ν . This is a generalization of [16, Lemma 4.5], and proved at stalks as follows. Let $s \in P^{\log}$. Let $s' \in P'^{\log}$ and $p \in P$ be its images. Let $B := F_p$. Let $K := \text{Hom}(C, \mathbf{Z})$, where C is the cokernel of $M_{S, f(p)}^{\text{gp}} / \mathcal{O}_{S, f(p)}^{\times} \rightarrow M_{P, p}^{\text{gp}} / \mathcal{O}_{P, p}^{\times}$. Then, K acts on the stalk $(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F)_s = \mathcal{O}_{P, s}^{\log} \otimes_{\mathcal{O}_{P, p}} B$ and we can see that the cohomology $H^q(K, (\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F)_s)$ is zero for $q > 0$ and is naturally isomorphic to $\mathcal{O}_{P', s'}^{\log} \otimes_{\mathcal{O}_{P, p}} B = (\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F)_{s'}$ for $q = 0$ (cf. the case where $n = 1$ of [7, Lemma (8.6.3.1) (1)]). Here we use the assumption of torsion-freeness and log injectiveness. Hence, we have (22) and

$$Rf_*^{\log}(\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F) \xrightarrow{\sim} Rf_*^{\log}(\mathcal{O}_P^{\log} \otimes_{\mathcal{O}_P} F).$$

The rest is to show that

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf'_* F \xrightarrow{\sim} Rf'_*{}^{\log}(\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F),$$

i.e., that 6.4 holds also for f' . Since $\mathcal{O}_{P'}^{\log} = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \mathcal{O}_P$ ([7, Lemma (3.6) (1)]), we have $\mathcal{O}_{P'}^{\log} \otimes_{\mathcal{O}_P} F = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F$. Further, since $P'^{\log} = S^{\log} \times_S P$, the proper base change theorem and the projection formula imply the desired result.

By 6.4, we have

$$\mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R^q f_* \omega_{P/S} \xrightarrow{\sim} R^q f_*{}^{\log} \omega_{P/S}^{\cdot, \log}. \quad (23)$$

Hence, by [7, Proposition (3.7) (3)], we get (21).

6.5. We prove the proposition 6.2 (i). First, by [7, Theorem (6.3)] and its proof, we know that the composite

$$\begin{aligned} \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log} \mathbf{Z} &\xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} \tau_*(\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log} \mathbf{Z}) \rightarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R\tau_* Rf_*^{\log} \mathcal{O}_S^{\log} \\ &\rightarrow \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} R\tau_* Rf_*^{\log} \omega_{P/S}^{\cdot, \log} = \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* R\tau_* \omega_{P/S}^{\cdot, \log} \\ &\xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* \omega_{P/S} \end{aligned} \quad (24)$$

is an isomorphism in the derived category.

Here, by (20), the second map is a quasi-isomorphism. Hence, the third one is also a quasi-isomorphism. Taking the q -th cohomology $\mathcal{H}^q$, we have the commutative diagram

$$\begin{array}{ccccc} \tau^* \mathcal{H}^q(\tau_*(Rf_*^{\log} \mathcal{O}_S^{\log})) & \xrightarrow{\sim} & \tau^* R^q \tau_* Rf_*^{\log} \mathcal{O}_S^{\log} & \xrightarrow{\sim} & \tau^* R^q f_* R\tau_* \omega_{P/S}^{\cdot, \log} \xleftarrow{\sim} \tau^* R^q f_* \omega_{P/S} \\ \downarrow \simeq & & \downarrow & & \downarrow \\ \mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} R^q f_*^{\log} \mathbf{Z} & \xleftarrow{\text{adj}} & \tau^* \tau_* R^q f_*^{\log} \mathcal{O}_S^{\log} & \longrightarrow & \tau^* \tau_* R^q f_*^{\log} \omega_{P/S}^{\cdot, \log}. \end{array}$$

Here, $\tau^*(-)$ means $\mathcal{O}_S^{\log} \otimes \tau^{-1}(-)$. By (20) and (21), the middle vertical arrow and the right vertical arrow are quasi-isomorphisms, respectively. Hence the bottom arrows are also quasi-isomorphisms. This completes the proof of 6.2 (i).

6.6. We prove 6.2 (ii). It is enough to show that the composite of quasi-isomorphisms in (24) coincides with the composite

$$\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} Rf_*^{\log} \mathbf{Z} \rightarrow Rf_*^{\log} \omega_{P/S}^{\cdot, \log} \xleftarrow{\sim} \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} Rf_* \omega_{P/S}, \quad (25)$$

where the latter morphism is a quasi-isomorphism by (23).

First we have the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log} \omega_{P/S}^{\cdot, \log} & = & \mathcal{O}_S^{\log} \otimes Rf_* R\tau_* \omega_{P/S}^{\cdot, \log} & \xleftarrow{\sim} & \mathcal{O}_S^{\log} \otimes Rf_* \omega_{P/S} \\ \downarrow \text{adj} & & \downarrow & & \downarrow \\ Rf_*^{\log} \omega_{P/S}^{\cdot, \log} & \xleftarrow{\text{adj}} & Rf_*^{\log} (\mathcal{O}_S^{\log} \otimes R\tau_* \omega_{P/S}^{\cdot, \log}) & \xleftarrow{\sim} & Rf_*^{\log} (\mathcal{O}_S^{\log} \otimes \omega_{P/S}), \end{array} \quad (26)$$

where the top row is nothing but the second and the third lines in (24) and the composite of the right vertical arrow and the bottom horizontal arrows coincides with the latter morphism in (25). Since the latter morphism in (25) is a quasi-isomorphism, the left vertical arrow in (26) is a quasi-isomorphism.

Second we have another commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_S^{\log} \otimes \tau_*(\mathcal{O}_S^{\log} \otimes Rf_*^{\log} \mathbf{Z}) & \xrightarrow{\sim} & \mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log} \mathcal{O}_S^{\log} & \xrightarrow{\sim} & \mathcal{O}_S^{\log} \otimes R\tau_* Rf_*^{\log} \omega_{P/S}^{\cdot, \log} \\ \simeq \downarrow & & \downarrow \text{adj} & & \downarrow \simeq \\ \mathcal{O}_S^{\log} \otimes Rf_*^{\log} \mathbf{Z} & \xlongequal{\quad} & \mathcal{O}_S^{\log} \otimes Rf_*^{\log} \mathbf{Z} & \longrightarrow & Rf_*^{\log} \omega_{P/S}^{\cdot, \log}, \end{array} \quad (27)$$

where the left vertical arrow and the top row are the ones which appeared in (24), the right vertical arrow is the same as the left vertical one in (26), and the bottom horizontal arrow is the former morphism in (25). This implies the desired compatibility and that the former morphism in (25) is a quasi-isomorphism, which completes the proof of 6.2.

7. Log de Rham complexes: absolute and relative

In this section, we prove 4.2.2. Let the notation be as in Section 4.

7.1. We define a homomorphism $s(Y) : f^{\log-1} \mathcal{O}_0^{\log} \rightarrow R\pi_* \mathbf{C}$ as the adjoint of the homomorphism of the constant sheaves $\pi^{-1} f^{\log-1} \mathcal{O}_0^{\log} = \mathbf{C} \left[\frac{\log t}{2\pi i} \right] \rightarrow \mathbf{C}$ on $Y_{\infty}^{\log}$ sending $\frac{\log t}{2\pi i}$ to 0. Note that $s(Y)$ is the pullback of $s(0)$ in the sense that

$s(Y)$ coincides with $f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{f^{\log-1}(s(0))} f^{\log-1}R\pi_*\mathbf{C} \xrightarrow{\sim} R\pi_*\mathbf{C}$, where the last quasi-isomorphism is by the facts that $Y^{\log}$ is a topological manifold and that $\pi : Y_{\infty}^{\log} \rightarrow Y^{\log}$ is locally trivial with discrete fibers. Note that we also have $R\pi_*\mathbf{C} = \pi_*\mathbf{C}$ on $Y^{\log}$ by the same reason.

We will reduce 4.2.2 to the following:

PROPOSITION 7.2. *The composite*

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \rightarrow R\tau_*\omega_{Y/0}^{\cdot,\log} \xleftarrow{\sim} \omega_{Y/0}^{\cdot,\log} \quad (28)$$

induced by the natural homomorphism $f^{\log-1}\mathcal{O}_0^{\log} \rightarrow \omega_{Y/0}^{\cdot,\log}$ coincides with the composite

$$\begin{aligned} R\tau_*f^{\log-1}\mathcal{O}_0^{\log} &\xrightarrow{R\tau_*(s(Y))} R(\tau\pi)_*\mathbf{C} \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \\ &\xleftarrow{\sim} \omega_Y[u] \xrightarrow{\sim} \omega_{Y/0}^{\cdot,\log} \end{aligned} \quad (29)$$

of the morphism $R\tau_*(s(Y))$ and the morphism (4) in 4.2.

7.3. We prove 7.2 in several steps.

First, we introduce an extended log de Rham complex. Let u be an indeterminate. Consider the complex $\omega_Y^{\cdot,\log}[u]$ on $Y^{\log}$ whose component of degree p is $\omega_Y^{p,\log} \otimes_{\mathbf{C}} \mathbf{C}[u]$, where $\mathbf{C}[u]$ is the constant sheaf, and whose derivation is defined by $u \mapsto \frac{d \log t}{2\pi i} \in \omega_Y^{1,\log}$. Let $e : f^{\log-1}\mathcal{O}_0^{\log} \rightarrow \omega_Y^{\cdot,\log}[u]$ be the morphism sending $\frac{\log t}{2\pi i}$ to $\frac{\log t}{2\pi i} - u$. Then, the composite

$$f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{e} \omega_Y^{\cdot,\log}[u] \xrightarrow{u \mapsto 0} \omega_{Y/0}^{\cdot,\log}$$

is nothing but the natural map $f^{\log-1}\mathcal{O}_0^{\log} \rightarrow \omega_{Y/0}^{\cdot,\log}$.

7.4. Next, we define a morphism of complexes

$$w : \pi^{-1}\omega_Y^{\cdot,\log}[u] \rightarrow \pi^{-1}\omega_Y^{\cdot,\log}$$

on $Y_{\infty}^{\log}$ by sending u to $\frac{\log t}{2\pi i} \in \mathcal{O}_Y^{\log}$. The well-definedness is shown as follows. For any p , let $u^m\eta \in \pi^{-1}\omega_Y^{p,\log}[u]$, where $\eta \in \pi^{-1}\omega_Y^{p,\log}$. Then,

$$d(u^m\eta) = mu^{m-1} \frac{d \log t}{2\pi i} \wedge \eta + u^m d\eta.$$

On the other hand, $w(u^m\eta) = \left(\frac{\log t}{2\pi i}\right)^m \eta$, and

$$d\left(\left(\frac{\log t}{2\pi i}\right)^m \eta\right) = m\left(\frac{\log t}{2\pi i}\right)^{m-1} \frac{d \log t}{2\pi i} \wedge \eta + \left(\frac{\log t}{2\pi i}\right)^m d\eta.$$

Hence, $wd(u^m\eta) = dw(u^m\eta)$.

The diagram

$$\begin{array}{ccc} \pi^{-1}f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{\pi^{-1}(e)} & \pi^{-1}\omega_Y^{\cdot,\log}[u] \\ \downarrow & & \downarrow w \\ \mathbf{C} & \xrightarrow{\sim} & \pi^{-1}\omega_Y^{\cdot,\log}, \end{array}$$

where the left vertical arrow is the morphism defined by $\frac{\log t}{2\pi i} \mapsto 0$, and the bottom quasi-isomorphism is by the log Poincaré lemma ([14, Theorem (3.8)]), is commutative. This is seen by $w\left((\pi^{-1}(e))\left(\frac{\log t}{2\pi i}\right)\right) = w\left(\frac{\log t}{2\pi i} - u\right) = \frac{\log t}{2\pi i} - \frac{\log t}{2\pi i} = 0$.

By taking the adjoint, we have a commutative diagram

$$\begin{array}{ccc} f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{e} & \omega_Y^{\cdot,\log}[u] \\ s(Y) \downarrow & & \downarrow \\ R\pi_*\mathbf{C} & \xrightarrow{\sim} & R\pi_*\pi^{-1}\omega_Y^{\cdot,\log}. \end{array}$$

7.5. The last commutativity implies that the morphism (29) in 7.2 coincides with

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{R\tau_*(e)} R\tau_*\omega_Y^{\cdot,\log}[u] \longrightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot,\log}[u] \xrightarrow{\sim} \omega_{Y/0}^{\cdot,\log}.$$

Hence the composite of the morphism (29) and the natural homomorphism $\omega_{Y/0}^{\cdot,\log} \xrightarrow{\sim} R\tau_*\omega_{Y/0}^{\cdot,\log}$ coincides with

$$\begin{aligned} R\tau_*f^{\log-1}\mathcal{O}_0^{\log} & \xrightarrow{R\tau_*(e)} R\tau_*\omega_Y^{\cdot,\log}[u] \longrightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot,\log}[u] \\ & \longrightarrow R\tau_*\omega_Y^{\cdot,\log}[u] \xrightarrow{u \mapsto 0} R\tau_*\omega_{Y/0}^{\cdot,\log}. \end{aligned}$$

On the other hand, 7.3 implies that the first morphism

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \rightarrow R\tau_*\omega_{Y/0}^{\cdot,\log}$$

in 7.2 coincides with the composite

$$R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{R\tau_*(e)} R\tau_*\omega_Y^{\cdot,\log}[u] \xrightarrow{u \mapsto 0} R\tau_*\omega_{Y/0}^{\cdot,\log}.$$

Therefore, 7.2 reduces to the statement that the composite

$$R\tau_*\omega_Y^{\cdot,\log}[u] \rightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\cdot,\log} \xleftarrow{\sim} \omega_Y^{\cdot,\log}[u] \rightarrow R\tau_*\omega_{Y/0}^{\cdot,\log}[u]$$

is the identity.

7.6. We prove the last statement. By [7, Proposition (3.7) (3)], $\omega_Y^{\cdot,\log}[u] \rightarrow R\tau_*\omega_Y^{\cdot,\log}[u]$ is a quasi-isomorphism. Hence, this statement is equivalent to that

the composite

$$\omega_Y[u] \rightarrow R\tau_*\omega_Y^{\log}[u] \rightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\log} \xleftarrow{\sim} \omega_Y[u]$$

is the identity. To see this, we consider the morphisms of complexes and the statement is deduced from the following two properties of the quasi-isomorphism $\omega_Y[u] \xrightarrow{\sim} R(\tau\pi)_*\pi^{-1}\omega_Y^{\log}$ by [6, Lemma 3.3].

- (i) It sends the indeterminate u to $\frac{\log t}{2\pi i} \in \mathcal{O}_Y^{\log}$.
- (ii) The composite $\omega_Y \rightarrow \omega_Y[u] \rightarrow R(\tau\pi)_*\pi^{-1}\omega_Y^{\log}$ is the one induced by the adjoint map.

This completes the proof of 7.2.

PROPOSITION 7.7. *We have the following.*

- (i) *The homomorphism*

$$Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \rightarrow Rf_*R\tau_*\omega_{Y/0}^{\log}$$

is a quasi-isomorphism.

- (ii) *The homomorphism*

$$Rf_*R\tau_*(s(Y)) : Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \rightarrow Rf_*R(\tau\pi)_*\mathbf{C}$$

is a quasi-isomorphism.

PROOF. (i) By 6.2 (ii), we have

$$Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{\sim} Rf_*^{\log}\omega_{Y/0}^{\log}.$$

By sending this by $R\tau_*$, we obtain (i).

(ii) By (i), the image by Rf_* of the morphism (28) in 7.2 is a quasi-isomorphism. Hence, by 7.2, the image by Rf_* of the morphism (29) in 7.2 is also a quasi-isomorphism. This implies (ii).

7.8. We prove 4.2.2 by the above proposition 7.2.

First, we prove that the homomorphism induced from

$$\begin{aligned} \mathcal{O}_0^{\log} \otimes Rf_*^{\log}\mathbf{C} &\xrightarrow{\sim} Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \xleftarrow{\text{adj}} \tau^{-1}R\tau_*Rf_*^{\log}f^{\log-1}\mathcal{O}_0^{\log} \\ &= \tau^{-1}Rf_*R\tau_*f^{\log-1}\mathcal{O}_0^{\log} \xrightarrow{\sim} \tau^{-1}Rf_*R\tau_*\omega_{Y/0}^{\log} \xleftarrow{\sim} \tau^{-1}Rf_*\omega_{Y/0}^{\log} \end{aligned} \quad (30)$$

by taking $\mathcal{H}^q$, taking the stalk at z and specializing by $\frac{\log t}{2\pi i} \mapsto 0$ coincides with (2). Here the first arrow in the second line of (30) is a quasi-isomorphism by 7.7 (i).

By (20) and (21), the composite of the three quasi-isomorphisms in 2.3 (i) coincides with the $\mathcal{H}^q$ of the homomorphism

$$\begin{aligned}
\mathcal{O}_0^{\log} \otimes Rf_*^{\log} \mathbf{C} &\xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes R\tau_*(\mathcal{O}_0^{\log} \otimes Rf_*^{\log} \mathbf{C}) \xrightarrow{\sim} \mathcal{O}_0^{\log} \otimes R\tau_* Rf_*^{\log} \omega_{Y/0}^{\log} \\
&= \mathcal{O}_0^{\log} \otimes Rf_* R\tau_* \omega_{Y/0}^{\log} \xleftarrow{\sim} \mathcal{O}_0^{\log} \otimes Rf_* \omega_{Y/0}^{\log}.
\end{aligned} \tag{31}$$

Hence, (2) coincides with the homomorphism induced from

$$\begin{aligned}
&\text{the stalk of } (\mathcal{O}_0^{\log} \otimes Rf_*^{\log} \mathbf{C}) \leftarrow R\tau_*(\mathcal{O}_0^{\log} \otimes Rf_*^{\log} \mathbf{C}) \\
&\xrightarrow{\sim} R\tau_* Rf_*^{\log} f^{\log-1} \mathcal{O}_0^{\log} \xrightarrow{\sim} R\tau_* Rf_*^{\log} \omega_{Y/0}^{\log} = Rf_* R\tau_* \omega_{Y/0}^{\log} \xleftarrow{\sim} Rf_* \omega_{Y/0}^{\log}
\end{aligned} \tag{32}$$

by taking $\mathcal{H}^q$ and specializing by $\frac{\log t}{2\pi i} \mapsto 0$. Since it is clear that (30) and (32) are compatible, we see that the homomorphism induced from (30) and (2) coincide, as desired.

Since the stalk of Rf_* of the composite $R(\tau\pi)_* \mathbf{C} \xrightarrow{\sim} R(\tau\pi)_* \pi^{-1} \omega_Y^{\log} \xleftarrow{\sim} \omega_Y[u] \xrightarrow{\sim} \omega_{Y/0}$ in (29) in 7.2 coincides with (5), 7.2 reduces 4.2.2 to the following claim.

CLAIM. The quasi-isomorphism

$$Rf_* R\tau_*(s(Y)) : Rf_* R(\tau\pi)_* \mathbf{C} \xleftarrow{\sim} Rf_* R\tau_* f^{\log-1} \mathcal{O}_0^{\log}$$

in 7.7 (ii) is compatible with the stalk of the homomorphism $\mathcal{O}_0^{\log} \otimes Rf_*^{\log} \mathbf{C} \xrightarrow{\sim} Rf_*^{\log} f^{\log-1} \mathcal{O}_0^{\log} \xleftarrow{\text{adj}} \tau^{-1} R\tau_* Rf_*^{\log} f^{\log-1} \mathcal{O}_0^{\log} = \tau^{-1} Rf_* R\tau_* f^{\log-1} \mathcal{O}_0^{\log}$.

To prove this claim, we use the following lemma.

LEMMA 7.9. *The diagram*

$$\begin{array}{ccc}
f^{\log-1} \mathcal{O}_0^{\log} & \longrightarrow & \pi_* \mathbf{C} \\
\downarrow & & \downarrow \\
(\pi i)_* (\pi i)^{-1} f^{\log-1} \mathcal{O}_0^{\log} & \longrightarrow & (\pi i)_* \mathbf{C}
\end{array}$$

is commutative, where the upper arrow is induced by the homomorphism $s(Y)$ in 7.1, the vertical arrows are the adjoint morphisms, and the bottom arrow is $(\pi i)_*$ of the morphism of constant sheaves $(\pi i)^{-1} f^{\log-1} \mathcal{O}_0^{\log} \rightarrow (\pi i)_* \mathbf{C}$ on Y_∞ defined by sending $\frac{\log t}{2\pi i}$ to 0.

PROOF. By taking the adjoint, the commutativity is reduced to that of

$$\begin{array}{ccc}
\pi^{-1} f^{\log-1} \mathcal{O}_0^{\log} & \xrightarrow{\frac{\log t}{2\pi i} \mapsto 0} & \mathbf{C} \\
\text{adj} \downarrow & & \downarrow \\
i_* i^{-1} \pi^{-1} f^{\log-1} \mathcal{O}_0^{\log} & \xrightarrow{\frac{\log t}{2\pi i} \mapsto 0} & i_* \mathbf{C}.
\end{array}$$

7.10. To prove the above claim, we consider the following commutative diagram:

$$\begin{array}{ccccc}
 Rf_*^{\log} f^{\log-1} \mathcal{O}_0^{\log} & \xleftarrow{\text{adj}} & \tau^{-1} R\tau_* Rf_*^{\log} f^{\log-1} \mathcal{O}_0^{\log} & \xrightarrow{\sim} & \tau^{-1} R\tau_* Rf_*^{\log} \pi_* \mathbf{C} \\
 \downarrow & & \downarrow & & \downarrow \\
 Rf_*^{\log} (\pi i)_* (\pi i)^{-1} f^{\log-1} \mathcal{O}_0^{\log} & \xleftarrow{\text{adj}} & \tau^{-1} R\tau_* Rf_*^{\log} (\pi i)_* (\pi i)^{-1} f^{\log-1} \mathcal{O}_0^{\log} & \longrightarrow & \tau^{-1} R\tau_* Rf_*^{\log} (\pi i)_* \mathbf{C},
 \end{array}$$

where the right square is induced from the commutative diagram in 7.9. By taking the stalk at z of this diagram, we get the following commutative diagram:

$$\begin{array}{ccccc}
 R\Gamma\left(Y_z^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xleftarrow{\text{adj}} & R\Gamma\left(Y^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xrightarrow{\sim} & R\Gamma(Y_{\infty}^{\log}, \mathbf{C}) \\
 \simeq \downarrow & & \downarrow & & \downarrow \simeq \\
 R\Gamma\left(Y_{\infty, z}^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xleftarrow{\text{id}} & R\Gamma\left(Y_{\infty, z}^{\log}, \mathbf{C}\left[\frac{\log t}{2\pi i}\right]\right) & \xrightarrow{\frac{\log t}{2\pi i} \mapsto 0} & R\Gamma(Y_{\infty, z}^{\log}, \mathbf{C}).
 \end{array}$$

This implies the claim and completes the proof of 4.2.2.

REMARK 7.11. It is plausible that the natural homomorphism

$$f^{\log-1} \mathcal{O}_0^{\log} \rightarrow \omega_{Y/0}^{\cdot, \log}$$

in 7.2 is a quasi-isomorphism, or more generally, that

$$f^{\log-1} \mathcal{O}_S^{\log} \rightarrow \omega_{P/S}^{\cdot, \log}$$

in 6.2 is a quasi-isomorphism. If this is the case, it makes proofs in Sections 6–7 considerably simpler.

8. Proof of main theorem

In this section, we prove the main theorem 2.7.

8.1. Let $f : Y \rightarrow 0$ be as in 2.7. We claim that the data

$$(H_{\mathbf{Q}}, W(N), H_{\mathcal{O}}, F, N), \quad (33)$$

where F is the Hodge filtration and N is the monodromy logarithm on $H_{\mathbf{Q}}$, are isomorphic to

$$(H^q(Y, K_{\mathbf{Q}}), W[q], H^q(Y, K_{\mathbf{C}}), F, N_K) \quad (34)$$

in Theorem 3.4. In fact, we have a filtered quasi-isomorphism (18)

$$(\omega_{Y/0}, F) \xrightarrow{\sim} (A_{\mathbf{C}}, F),$$

where F on the source is defined by $\omega_{Y/0}^{\geq r}$, and the morphism $A_C \rightarrow K_C$ in (15) such that Theorem 5.21 and Theorem 5.29 in [5] enable us to identify

$$(H_\emptyset, F) \simeq (H^q(Y, A_C), F) \simeq (H^q(Y, K_C), F) \quad (35)$$

as filtered $\mathbf{C}$ -vector spaces. The $\mathbf{Q}$ -structures coincide by 4.1.1, which is now proved by the results in Sections 5 and 7. The coincidence of N on $H_{\mathbf{Q}}$ and N_K on $H^q(Y, K_{\mathbf{Q}})$ is proved in 5.17. Then the weight filtrations coincide because they are the monodromy filtrations of the compatible operators N and N_K .

8.2. Now we prove the goodness as follows. By 6.1, the first condition (i) of the goodness is always satisfied. The second condition is by the coincidence of the Hodge filtration (35) and the fact that (A_C, F) is a part of a CMHC ([5, Theorem 5.21]). The third condition of the goodness, that is, that the data form an LH, is proved as follows. The Griffiths transversality (2.1 (iii)) holds because the construction in [5] satisfies the Griffiths transversality (a part of the definition of polarized MHS in [2, Definition (2.26)]). Condition 2.1 (ii) also holds because the construction in [5] is a MHS. Then, [2, Corollary (3.13)] implies Condition 2.1 (i).

9. Log Picard varieties and log Albanese varieties

In this section, we apply the main result 2.7 to the theory of log Picard and log Albanese varieties.

9.1. First we recall the definition of log Picard variety and log Albanese variety briefly. See [10] for the details.

Let S be an fs log analytic space. Let $\mathcal{A}_S$ be the category of log complex tori over S and let $\mathcal{H}_S$ be the category of log Hodge structures H on S of weight -1 satisfying $F^{-1}H_\emptyset = H_\emptyset$ and $F^1H_\emptyset = 0$. Then we have a category equivalence

$$\mathcal{H}_S \simeq \mathcal{A}_S.$$

Let $P \rightarrow S$ be as in 6.2. Assume that it is good for $\mathcal{H}^1$ in the sense of [10, 7.1]. Then the log Picard variety $A_{P/S}^*$ is the log complex torus corresponding to $\mathcal{H}^1(P)(1)$ and the log Albanese variety $A_{P/S}$ is the log complex torus corresponding to the dual log Hodge structure of $\mathcal{H}^1(P)$. Here $\mathcal{H}^1(P)$ is the log Hodge structure underlain by $R^1f_*^{\log}\mathbf{Z}$ which is given by the definition of the goodness.

9.2. We prove that a projective log deformation $Y \rightarrow 0$ is good even in the sense of [10, 7.1] so that there are well-defined log complex tori $A_{Y/0}^*$ and $A_{Y/0}$.

For this, we prove that our definition of goodness coincides with that in [10]. As in 2.3, let $Y \rightarrow 0$ be a proper, separated and log smooth morphism of fs log analytic spaces such that for any $y \in Y$, the cokernel of $M_0^{\text{gp}}/\mathcal{O}_0^\times \rightarrow M_{Y,y}^{\text{gp}}/\mathcal{O}_{Y,y}^\times$ is torsion-free.

We do not recall the definition of the goodness in [10] here, but recall that its main difference from the goodness in Section 2 of this paper lies in that, in [10], the sheaves are all considered on the big sites.

Let $f : P \rightarrow S$ be as in 6.2. Then, any (fs) base change of f also satisfies the assumption of 6.2 so that the conclusion of 6.2 is satisfied also for any base changed morphism. This means that f satisfies the condition (i) in [10, 7.1]. (Actually, the modified one as explained in 2.4.)

Further, [11, Proposition 5.1] (the proper base change theorem in log Betti cohomology) implies that the construction of $R^q f^{\log} \mathbf{Z}$ is compatible with any base change. This means that the former part of the condition (iii) in [10, 7.1] is satisfied.

Thus, in particular, $Y \rightarrow 0$ as above always satisfies (i) and the former part of (iii) in [10, 7.1].

Next, in order to see that the condition (ii) for $Y \rightarrow 0$ in [10, 7.1] is equivalent to our (ii) in 2.3, it is enough to show that the constructions of the two maps in 2.3 (ii) are compatible with any base change. (Note that the injection there is a split injection.) This is by the fact that the construction of the relative differential sheaf is compatible with any (fs) base change (cf. [12, 1.7 and 3.12]) and by the usual flat base change theorem.

Finally, the latter part of the condition (iii) in [10, 7.1] is the same as our 2.3 (ii). This completes the proof of the equivalence of our definition of goodness and that in [10, 7.1]. In particular, our main theorem 2.7 implies that a projective log deformation $Y \rightarrow 0$ is good even in the sense of [10, 7.1].

9.3. There is another definition of the goodness in [10, 7.2]. For an f as in 6.2 (especially, for a projective log deformation), both goodness in [10, 7.1] and in [10, 7.2] coincide and the quasi-isomorphisms given by the first conditions in both definitions also coincide. This is a consequence of 6.2 (ii).

9.4. To discuss the applications, we further introduce notation.

Let (fs/ S) be the category of fs log analytic spaces over S endowed with the usual topology. Recall that a log complex torus over S is a sheaf of abelian groups on (fs/ S).

Let (fs/ S)^{log} be the site of pairs (U, T) , where $T \in (\text{fs}/S)$ and U is an open set of $T^{\log}$, defined in [9, 3.1.6] (cf. [10, 3.2]). For a sheaf F on (fs/ S)^{log}, we denote by $\tau_* F$ the sheaf on (fs/ S) defined by $(\tau_* F)(T) = F((T^{\log}, T))$ for $T \in (\text{fs}/S)$.

Let $P \in (\text{fs}/S)$ and $q \in \mathbf{Z}$.

For a sheaf F of abelian groups on (fs/P) , let $\mathcal{H}^q(P, F)$ be the sheaf of abelian groups on (fs/S) whose restriction on any $T \in (\text{fs}/S)$ is $R^q(P \times_S T \rightarrow T)_*(F_{P \times_S T})$. Here $F_{P \times_S T}$ is the restriction of F on the small site of $P \times_S T$.

For a sheaf F of abelian groups on $(\text{fs}/P)^{\log}$, let $\mathcal{H}^q(P^{\log}, F)$ (denoted by $\mathcal{H}^q(\tau^{-1}(P), F)$ in [10]) be the sheaf of abelian groups on $(\text{fs}/S)^{\log}$ whose restriction on the small site of $(T^{\log}, T)$ for any $T \in (\text{fs}/S)$ is $R^q(P \times_S T \rightarrow T)_*^{\log}(F_{(P \times_S T)^{\log}})$. Here $F_{(P \times_S T)^{\log}}$ is the restriction of F on the small site of $(P \times_S T)^{\log}$.

Let $\mathbf{G}_m$ (resp. $\mathbf{G}_{m, \log}$) be the sheaf of abelian groups on (fs/P) defined by $\mathbf{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times)$ (resp. $\mathbf{G}_{m, \log}(T) = \Gamma(T, M_T^{\text{gp}})$) for $T \in (\text{fs}/P)$.

Now we apply Section 8 of [10] on log Picard varieties. Note that the assumption 8.1 of [10] is satisfied (cf. the remark after it). Then, *ibid.* 8.2 and 8.3 imply the following.

THEOREM 9.5. *Let $Y \rightarrow 0$ be a projective log deformation.*

(i) *Let*

$$\begin{aligned}\mathcal{H}^1(Y, \mathbf{G}_m)_0 &= \text{Ker}(\mathcal{H}^1(Y, \mathbf{G}_m) \rightarrow \mathcal{H}^2(Y, \mathbf{Z})), \\ \mathcal{H}^1(Y, \mathbf{G}_{m, \log})_0 &= \text{Ker}(\mathcal{H}^1(Y, \mathbf{G}_{m, \log}) \rightarrow \tau_* \mathcal{H}^2(Y^{\log}, \mathbf{Z})).\end{aligned}$$

Then we have canonical embeddings

$$\mathcal{H}^1(Y, \mathbf{G}_m)_0 \subset A_{Y/0}^* \subset \mathcal{H}^1(Y, \mathbf{G}_{m, \log})_0.$$

(ii) *We have a canonical isomorphism*

$$\mathcal{H}^1(Y, \mathbf{G}_{m, \log}) / \mathcal{H}^1(Y, \mathbf{G}_{m, \log})_0 \simeq \mathcal{H}om(\mathbf{Z}, \mathcal{H}^2(Y)(1)).$$

Here $\mathcal{H}^2(Y)$ is the log Hodge structure underlain by $R^2(Y^{\log} \rightarrow 0^{\log})_* \mathbf{Z}$ which is given by the definition of the goodness, and $\mathcal{H}om$ is the $\mathcal{H}om$ for log Hodge structures.

We remark that in (i), more precisely, [10, Theorem 8.2] describes the groups $\mathcal{H}^1(Y, \mathbf{G}_m)_0$, $\mathcal{H}^1(Y, \mathbf{G}_{m, \log})_0$, and the quotients $A_{Y/0}^* / \mathcal{H}^1(Y, \mathbf{G}_m)_0$ and $\mathcal{H}^1(Y, \mathbf{G}_{m, \log})_0 / A_{Y/0}^*$ in terms of the log Hodge structure $\mathcal{H}^1(Y)(1)$. We omit here the details.

COROLLARY 9.6. *We have an exact sequence*

$$0 \rightarrow \mathcal{E}xt^1(A_{Y/0}, \mathbf{G}_{m, \log}) \rightarrow \mathcal{H}^1(Y, \mathbf{G}_{m, \log}) \rightarrow \tau_* \mathcal{H}^2(Y^{\log}, \mathbf{Z}) \rightarrow \mathcal{H}^2(Y, \mathcal{O}_Y).$$

Next, we apply Section 10 of [10] on log Albanese varieties.

PROPOSITION 9.7. *Let $Y \rightarrow 0$ be a connected projective log deformation. Then, for a log complex torus B over 0 , the following sequence is exact.*

$$0 \rightarrow B(0) \rightarrow B(Y) \rightarrow \mathrm{Hom}(A_{Y/0}, B).$$

9.8. Finally we discuss some problems.

PROBLEM 1. Prove that the last arrow $B(Y) \rightarrow \mathrm{Hom}(A_{Y/0}, B)$ is surjective so that each section e of $Y/0$ induces the *log Albanese map* $\psi_e : Y \rightarrow A_{Y/0}$.

To prove this, it is enough to show that the extension “ $\psi_e(a)$ ” of log Hodge structures in the page 177 of [10] is admissible.

PROBLEM 2 (cf. [10, Problem 10.6]). Prove the following: Let $Y/0$ be a projective log deformation. Then we have the inclusion

$$\mathcal{H}^1(P, \mathbf{G}_{m, \log})_0 \cap \mathrm{Image}(\mathcal{H}^1(P, \mathbf{G}_m) \rightarrow \mathcal{H}^1(P, \mathbf{G}_{m, \log})) \subset A_{Y/0}^*.$$

When this statement is valid, the inclusion map should be called the *log Abel–Jacobi map*.

More generally, we can ask the following.

PROBLEM 3 (cf. [10, 10.7]). Define a *log Abel–Jacobi map of degree r* whose target is the *log intermediate Jacobian* $\mathcal{E}xt^1(\mathbf{Z}, H^{2r-1}(Y)(r))$ of degree r , where $\mathcal{E}xt^1$ is taken in the category of log mixed Hodge structures.

Problem 1 should be the case of Problem 3 where $r = \dim Y$, and Problem 2 is the case $r = 1$.

Appendix

Because we could not find an appropriate reference, we give a sketch of the proof of the following basic statement, though it is not used in the text.

THEOREM A.1. *Let $f : X \rightarrow S$ be a proper, separated, smooth and Kähler morphism of complex analytic spaces. Then the triple*

$$(R^q f_* \mathbf{Z}, (R^q f_* \Omega_{X/S}, F), \iota)$$

is a variation of Hodge structures of weight q for every integer q , where $\Omega_{X/S}$ denotes the relative de Rham complex, F the decreasing filtration induced by the stupid filtration on $\Omega_{X/S}$, and $\iota : R^q f_ \mathbf{Z} \rightarrow R^q f_* \Omega_{X/S}$ the morphism induced*

by the canonical morphism $\mathbf{Z} \hookrightarrow \mathcal{O}_X \rightarrow \Omega_{X/S}$. Moreover, the Hodge to de Rham spectral sequence degenerates at E_1 -terms.

PROOF. By the result of Siebenmann [19, Corollary 6.14] (cf. [14, Proposition 3.5, Remark 3.5.1]) $R^q f_* \mathbf{Z}$ is a locally constant abelian sheaf of finite rank on S . By the relative Poincaré lemma [4, Théorème 2.23 (ii)] and by the projection formula, the morphism ι induces an isomorphism $\mathcal{O}_S \otimes R^q f_* \mathbf{Z} \xrightarrow{\sim} R^q f_* \Omega_{X/S}$. Thus $R^q f_* \Omega_{X/S}$ is a locally free $\mathcal{O}_S$ -module of finite rank and compatible with any base change.

Now we take a point $s \in S$ and set $A = \mathcal{O}_{S,s}$ for a while. We denote by $\mathfrak{m}$ the maximal ideal of A . By [1, Théorème 4.1] (cf. [17, Section 5]), there exists a complex K of $\mathcal{O}_{S,s}$ -modules satisfying the following:

- K is bounded from the above and the below.
- K^n is a free A -module of finite rank for every n .
- For a coherent $\mathcal{O}_S$ -module $\mathcal{F}$, we have a functorial isomorphism

$$Rf_*(\Omega_{X/S}^p \otimes f^* \mathcal{F})_s \simeq K \otimes \mathcal{F}_s$$

in the derived category.

Therefore we have

$$\begin{aligned} l_{A/\mathfrak{m}^{n+1}}(R^q f_*(\Omega_{X/S}^p \otimes A/\mathfrak{m}^{n+1})_s) \\ &= l_{A/\mathfrak{m}^{n+1}}(H^q(K \otimes A/\mathfrak{m}^{n+1})) \\ &\leq l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim H^q(K \otimes A/\mathfrak{m}) \\ &= l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim R^q f_*(\Omega_{X/S}^p \otimes A/\mathfrak{m})_s \end{aligned} \tag{36}$$

for every $n \geq 0$ by the case $k = 0$ of [3, Corollaire (3.4)], where $l_{A/\mathfrak{m}^{n+1}}(-)$ denotes the lengths of the finite modules over $A/\mathfrak{m}^{n+1}$.

On the other hand, we have the isomorphism

$$A/\mathfrak{m}^{n+1} \otimes (R^q f_* \mathbf{Z})_s \simeq R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m}^{n+1})_s$$

for every n, q by the relative Poincaré lemma and by the projection formula. Thus we obtain

$$l_{A/\mathfrak{m}^{n+1}}(R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m}^{n+1})_s) = l_{A/\mathfrak{m}^{n+1}}(A/\mathfrak{m}^{n+1}) \dim R^q f_*(\Omega_{X/S} \otimes A/\mathfrak{m})_s$$

for every n, q . By considering the Hodge to de Rham spectral sequences for $A/\mathfrak{m}^{n+1}$ and $A/\mathfrak{m}$, we conclude that the equality holds in (36) for every n, p, q and that the Hodge to de Rham spectral sequences for $A/\mathfrak{m}^{n+1}$ and $A/\mathfrak{m}$

degenerate at E_1 -terms as in [3, Section 5]. By [3, Corollaire (3.4), Remarque (3.3.3)] again, $H^q(K)$ is a free A -module and the canonical morphism $H^q(K) \otimes M \rightarrow H^q(K \otimes M)$ is an isomorphism for every A -module M and for every q . Therefore $R^q f_* \Omega_{X/S}^p$ is a locally free $\mathcal{O}_S$ -module of finite rank and compatible with any base change for every p, q . Moreover the Hodge to de Rham spectral sequence degenerates at E_1 -terms by the local freeness and the base change property of $R^q f_* \Omega_{X/S}^p$ and $R^q f_* \Omega_{X/S}$, and by the E_1 -degeneration of the Hodge to de Rham spectral sequence for $A/\mathfrak{m}$. Once we know the base change property above, it is easy to see that the pair

$$((R^q f_* \mathbf{Z})_s, (R^q f_* \Omega_{X/S}, F) \otimes \mathbf{C}(s))$$

equipped with the isomorphism induced by ι is a Hodge structure of weight q .

What remains to check is the Griffiths transversality. For this purpose, we consider Ω_X and Ω_S . We denote by $G^r \Omega_X^p$ the image of the morphism $\Omega_S^r \otimes \Omega_X^{p-r} \rightarrow \Omega_X^p$ induced by the wedge product. Then G defines a finite decreasing filtration on Ω_X . On the other hand, the stupid filtration on Ω_S is denoted by G for a while. By the smoothness of f , we have

$$\mathrm{gr}_G^r \Omega_X \simeq f^* \Omega_S^r \otimes \Omega_{X/S}[-r] = f^{-1} \Omega_S^r \otimes \Omega_{X/S}[-r]$$

for every r . Therefore the canonical morphism $f^{-1} \Omega_S \rightarrow \Omega_X$ is a filtered quasi-isomorphism by the relative Poincaré lemma and by the flatness of $\Omega_{X/S}^p$ over $f^{-1} \mathcal{O}_S$. Then we have

$$\begin{aligned} \Omega_S^p \otimes R^q f_* \mathbf{C} &\simeq R^q f_* f^{-1} \Omega_S^p \simeq E_1^{p,q}(Rf_* f^{-1} \Omega_S, G) \\ &\simeq E_1^{p,q}(Rf_* \Omega_X, G) \simeq \Omega_S^p \otimes R^q f_* \Omega_{X/S} \end{aligned}$$

by the projection formula and by the base change property of $R^q f_* \Omega_{X/S}$. Moreover, the morphism of E_1 -terms

$$\Omega_S^p \otimes R^q f_* \Omega_{X/S} \rightarrow \Omega_S^{p+1} \otimes R^q f_* \Omega_{X/S}$$

is identified with the morphism

$$d \otimes \mathrm{id} : \Omega_S^p \otimes R^q f_* \mathbf{C} \rightarrow \Omega_S^{p+1} \otimes R^q f_* \mathbf{C}$$

under this identification. Thus we can check the Griffiths transversality in the usual way.

REMARK A.2. Note that S is not necessarily reduced in the theorem above.

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Nonautonomous differential equations and Lipschitz evolution operators in Banach spaces

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ABSTRACT. A new class of Lipschitz evolution operators is introduced and a characterization of continuous infinitesimal generators of such evolution operators is given. It is shown that a continuous mapping A from a subset Ω of $[a, b) \times X$ into X , where $[a, b)$ is a real half-open interval and X is a real Banach space, is the infinitesimal generator of a Lipschitz evolution operator if and only if it satisfies a sub-tangential condition, a general type of quasi-dissipative condition with respect to a metric-like functional and a connectedness condition. An application of the results to the initial value problem for the quasilinear wave equation with dissipation is also given.

1. Introduction and main theorems

Throughout this paper, $\mathbf{R}$ denotes the set of all real numbers. Let X be a real Banach space with norm $\|\cdot\|$. For a subset Q of $\mathbf{R} \times X$, $Q(t)$ denotes the section of Q at $t \in \mathbf{R}$, that is, $Q(t) = \{x \in X; (t, x) \in Q\}$.

Let $[a, b)$ be a subinterval of $\mathbf{R}$ and Ω a subset of $[a, b) \times X$ such that $-\infty < a < b \leq \infty$ and $\Omega(t) \neq \emptyset$ for $t \in [a, b)$. Let A be a continuous mapping from Ω to X . Given $(\tau, z) \in \Omega$, we consider the following initial value problem:

$$(IVP; \tau, z) \quad \begin{cases} u'(t) = A(t, u(t)) & \text{for } \tau \leq t < b, \\ u(\tau) = z. \end{cases}$$

Suppose that the problem $(IVP; \tau, z)$ has a unique solution $u(\cdot)$ on $[\tau, b)$ for every $(\tau, z) \in \Omega$. Defining $U(t, \tau)z = u(t)$, we have the following properties from the uniqueness of solutions:

$$(E1) \quad U(\tau, \tau)z = z \text{ and } U(t, s)U(s, \tau)z = U(t, \tau)z \text{ for } z \in \Omega(\tau) \text{ and } a \leq \tau \leq s \leq t < b.$$

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Set $\mathcal{A} = \{(t, \tau); a \leq \tau \leq t < b\}$. Usually, we have also the following properties from the continuous dependence of solutions on the initial data $(\tau, z) \in \Omega$:

- (E2) Let $(t, \tau) \in \mathcal{A}$, $z \in \Omega(\tau)$, $(t_n, \tau_n) \in \mathcal{A}$ and $z_n \in \Omega(\tau_n)$ for $n = 1, 2, \dots$. If $(t_n, \tau_n) \rightarrow (t, \tau)$ and $z_n \rightarrow z$ as $n \rightarrow \infty$, then $U(t_n, \tau_n)z_n \rightarrow U(t, \tau)z$ as $n \rightarrow \infty$.

By an *evolution operator* on Ω , we mean a family $\{U(t, \tau)\}_{(t, \tau) \in \mathcal{A}}$ of operators $U(t, \tau) : \Omega(\tau) \rightarrow \Omega(t)$ satisfying (E1) and (E2). Such a family $\{U(t, \tau)\}_{(t, \tau) \in \mathcal{A}}$ is called a *Lipschitz evolution operator* on Ω , if the following additional condition is satisfied:

- (E3) There exist a number $L \geq 1$ and a continuous function $\omega : [a, b) \rightarrow [0, \infty)$ such that

$$\|U(t, \tau)x - U(t, \tau)y\| \leq L \exp\left(\int_{\tau}^t \omega(\theta) d\theta\right) \|x - y\|$$

for $x, y \in \Omega(\tau)$ and $(t, \tau) \in \mathcal{A}$.

The main purpose of this paper is to establish the conditions on the continuous mapping A which are necessary and sufficient to guarantee the existence of the Lipschitz evolution operator associated with A . The obtained results extend that of Kobayashi and Tanaka in [8] concerning the autonomous case where A is independent of t . In particular, a type of generalized quasi-dissipativity condition on A with respect to a metric-like functional is shown to be necessary for the existence of the Lipschitz evolution operator. Sufficient conditions on A for the existence of evolution operators have been studied by many authors and this paper is related with the works of Iwamiya [4], Kato [5], [6], Kenmochi and Takahashi [7], Lakshmikantham, Mitchell and Mitchell [10], Martin [11], [12], [13], Murakami [15], Pavel and Vrabie [19], Pavel [18] and Cârjă, Necula and Vrabie [22]. Several types of generalized quasi-dissipativity conditions on A are introduced and investigated in [15], [12], [10], [6], [20] and [2]. Such a kind of generalized quasi-dissipativity conditions was first found by Okamura [17] as a uniqueness criteria for ordinary differential equations. See [1] or [24]. Our results extend the most of them. As in [7], [6] and [4], the domain Ω is allowed to be genuinely noncylindrical and the subtangential condition, which was first found by Nagumo [16], is used to construct approximate solutions to $(\text{IVP}; \tau, z)$. The advantage of these assumptions is illustrated by an application of the results to the initial value problems for nonlinear wave equations.

Let $J \subset [a, b)$ be a subinterval of the form $[\tau, c]$ or $[\tau, c)$. An X -valued continuous function $u : J \rightarrow X$ is called a *solution to $(\text{IVP}; \tau, z)$ on J* , if $u(\tau) = z$, $(t, u(t)) \in \Omega$ for $t \in J$, u is differentiable on J and $u'(t) = A(t, u(t))$ for $t \in J$. A solution to $(\text{IVP}; \tau, z)$ on $[\tau, b)$ is called a *global solution*.

Let $d(x, D)$ denote the distance from $x \in X$ to $D \subset X$, i.e., $d(x, D) = \inf\{\|x - y\|; y \in D\}$. We consider the following conditions.

- ($\Omega 1$) A is continuous on Ω .
- ($\Omega 2$) If $(t_n, x_n) \in \Omega$, $t_n \uparrow t \in [a, b]$ in $\mathbf{R}$ and $x_n \rightarrow x$ in X as $n \rightarrow \infty$, then $(t, x) \in \Omega$.
- ($\Omega 3$) $\liminf_{h \downarrow 0} h^{-1} d(x + hA(t, x), \Omega(t + h)) = 0$ for $(t, x) \in \Omega$.
- ($\Omega 4$) There exists a functional $V : [a, b] \times X \times X \rightarrow [0, \infty)$ satisfying the following properties ($V 1$)–($V 4$) and a continuous function $\omega : [a, b] \rightarrow [0, \infty)$ such that

$$D_+ V(t, x, y)(A(t, x), A(t, y)) \leq \omega(t) V(t, x, y)$$

for $x, y \in \Omega(t)$ and $t \in [a, b]$. Here, for $(t, x, y) \in [a, b] \times X \times X$ and $(\zeta, \eta) \in X \times X$,

$$D_+ V(t, x, y)(\zeta, \eta) = \liminf_{h \downarrow 0} \frac{1}{h} (V(t + h, x + h\zeta, y + h\eta) - V(t, x, y)),$$

where the values ∞ and $-\infty$ are not excluded.

- ($V 1$) There exists a number $L > 0$ such that $|V(t, x, y) - V(t, \hat{x}, \hat{y})| \leq L(\|x - \hat{x}\| + \|y - \hat{y}\|)$ for $(x, y), (\hat{x}, \hat{y}) \in X \times X$ and $t \in [a, b]$.
- ($V 2$) $V(t, x, x) = 0$ for $t \in [a, b]$ and $x \in \Omega(t)$.
- ($V 3$) If $\{t_n\}$ is a sequence in $[a, b]$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \geq 1$, $t_n \rightarrow t \in [a, b]$ and $(x_n, y_n) \rightarrow (x, y) \in \Omega(t) \times \Omega(t)$ as $n \rightarrow \infty$, then $V(t, x, y) \leq \liminf_{n \rightarrow \infty} V(t_n, x_n, y_n)$.
- ($V 4$) If $\{t_n\}$ is a sequence in $[a, b]$ and $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ for $n \geq 1$, $t_n \rightarrow t \in [a, b]$ and $V(t_n, x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$, then $\|x_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.
- ($\Omega 5$) For any $(\tau, z) \in \Omega$, there exists a connected component C of Ω such that $(\tau, z) \in C$ and $C(t) \neq \emptyset$ for $t \in (\tau, b)$.

REMARK 1. Condition ($V 1$) with ($V 2$) implies the following:

$$|V(t, x, y)| \leq L\|x - y\| \quad \text{for } (x, y) \in \Omega(t) \times \Omega(t) \text{ and } t \in [a, b].$$

The following are our main theorems.

THEOREM 1. Let A be a mapping from Ω into X such that conditions ($\Omega 1$)–($\Omega 4$) are satisfied. Let C be a connected component of Ω and set $d = \sup\{t \in [a, b]; C(t) \neq \emptyset\}$. Then the following assertions hold true:

- (i) For $(\tau, z) \in C$, (IVP; τ, z) has a unique solution $u(t; \tau, z)$ on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution.

(ii) For $z, \hat{z} \in C(\tau)$ and $t \in [\tau, d)$,

$$V(t, u(t; \tau, z), u(t; \tau, \hat{z})) \leq \exp\left(\int_{\tau}^t \omega(\theta) d\theta\right) V(\tau, z, \hat{z}).$$

THEOREM 2. *Let A be a mapping from Ω into X such that $(\Omega 1)$ and $(\Omega 2)$ are satisfied. Then there exists a Lipschitz evolution operator $\{U(t, \tau)\}_{(t, \tau) \in \Delta}$ on Ω such that $u(t) := U(t, \tau)z$ is a global solution to $(\text{IVP}; \tau, z)$ for any $(\tau, z) \in \Omega$ if and only if conditions $(\Omega 3)$ – $(\Omega 5)$ are satisfied, where condition $(V 4)$ is replaced by the following condition:*

$(V 4)'$ For any $t \in [a, b)$ and $x, y \in \Omega(t)$, $\|x - y\| \leq V(t, x, y)$.

Theorem 1 consists of the uniqueness and local existence of solutions to initial value problems $(\text{IVP}; \tau, z)$ and the global existence theorem as well as the continuous dependence of solutions on initial data. They are discussed in Sections 2 and 3 respectively. The proof of Theorem 2 is given in Section 4. An application of our results to the initial value problem for quasi-linear wave equations is given in Section 5.

2. Uniqueness and local existence of solutions

In this section, we construct the solutions to the initial value problem $(\text{IVP}; \tau, z)$. We assume that conditions $(\Omega 1)$ – $(\Omega 4)$. The following proposition ensures the uniqueness of solutions.

PROPOSITION 1. *Let $[\tau, c) \subset [a, b)$ and $z_i \in \Omega(\tau)$ for $i = 1, 2$. Let u_i be solutions to $(\text{IVP}; \tau, z_i)$ on $[\tau, c)$, for $i = 1, 2$, respectively. Then*

$$V(t, u_1(t), u_2(t)) \leq \exp\left(\int_{\tau}^t \omega(s) ds\right) V(\tau, z_1, z_2)$$

for $t \in [\tau, c)$. In particular, if $z_1 = z_2$, then $u_1(t) = u_2(t)$ for $t \in [\tau, c)$.

PROOF. Set $w(t) = V(t, u_1(t), u_2(t))$ for $t \in [\tau, c)$. From $(V 3)$ we see that w is lower semi-continuous on $[\tau, c)$. Let $t \in [\tau, c)$ and $h \in (0, c - t)$. From $(V 1)$ it follows that

$$\begin{aligned} & (w(t+h) - w(t))/h - (V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) \\ & \quad + hA(t, u_2(t))) - V(t, u_1(t), u_2(t)))/h \\ & \leq |V(t+h, u_1(t+h), u_2(t+h)) \\ & \quad - V(t+h, u_1(t) + hA(t, u_1(t)), u_2(t) + hA(t, u_2(t)))|/h \end{aligned}$$

$$\begin{aligned} &\leq L(\|u_1(t+h) - u_1(t) - hA(t, u_1(t))\|/h \\ &\quad + \|u_2(t+h) - u_2(t) - hA(t, u_2(t))\|/h). \end{aligned}$$

Taking the inferior limit as $h \downarrow 0$ yields

$$\liminf_{h \downarrow 0} (w(t+h) - w(t))/h \leq D_+ V(t, u_1(t), u_2(t))(A(t, u_1(t)), A(t, u_2(t))).$$

From (Ω4) we have $D_+ w(t) \leq \omega(t)w(t)$, where $D_+ w(t)$ denotes the lower right derivative of $w(t)$. Therefore, we see that the function

$$t \rightarrow \exp\left(-\int_{\tau}^t \omega(s)ds\right)w(t)$$

is lower semicontinuous on $[\tau, c]$ and $D_+(\exp(-\int_{\tau}^t \omega(s)ds)w(t)) \leq 0$ for $t \in [\tau, c]$. By [3, Lemma 6.3], we have $w(t) \leq \exp(\int_{\tau}^t \omega(s)ds)w(\tau)$ for $t \in [\tau, c]$. Refer to [9] or [21] for the same kind of differential inequalities. $\square$

For each $(t, x) \in \mathbf{R} \times X$ and $r > 0$, we define $S_r(t, x) = \{(s, y) \in \mathbf{R} \times X; |s - t| < r, \|y - x\| < r\}$. We need the following lemmas which are proved in [7] without using condition (Ω4).

LEMMA 1 ([7, Lemma 1]). *Let $(t, x) \in \Omega$ and $\eta > 0$. Let $r > 0$ be a number such that $\|A(s, y) - A(t, x)\| \leq \eta$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $M > 0$ be a number such that $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Set $h_0 = \min\{r, r/M, b - t\}$. Then*

$$d(x + hA(t, x), \Omega(t+h)) \leq h\eta \quad \text{for } h \in (0, h_0).$$

LEMMA 2 ([7, Lemma 2]). *Let $(t, x) \in \Omega$ and $\varepsilon \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and such that $\|A(s, y) - A(t, x)\| \leq \varepsilon/3$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $h \in (0, r/(M+1))$. Let $\{s_k\}_{k=0}^n$ be a partition of $[t, t+h] : t = s_0 < s_1 < \dots < s_n = t+h$. Then there exists a sequence $\{y_k\}_{k=0}^n$ of elements in X such that*

- (i) $y_0 = x$ and $(s_k, y_k) \in \Omega$ for $0 \leq k \leq n$;
- (ii) $\|y_k - x\| \leq (M + \varepsilon)(s_k - t)$ for $0 \leq k \leq n$;
- (iii) $\|y_{k-1} + (s_k - s_{k-1})A(s_{k-1}, y_{k-1}) - y_k\| \leq \varepsilon(s_k - s_{k-1})$ for $1 \leq k \leq n$.

We also need the following lemma.

LEMMA 3. *Let $(t, x) \in \Omega$ and $\varepsilon \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $\sigma \in (0, r/(M+1))$. Then the following assertions hold true:*

(i) If a sequence $\{(s_i, y_i)\}_{i=0}^n$ in Ω satisfies

$$t = s_0 < s_1 < \cdots < s_n \leq t + \sigma, \quad (2.1)$$

$$\begin{aligned} \|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| &\leq \varepsilon(s_i - s_{i-1}) \\ \text{for } 1 \leq i \leq n, \text{ where } y_0 &= x, \end{aligned} \quad (2.2)$$

then

$$\begin{aligned} \|y_i - y_j\| &\leq (M + \varepsilon)(s_i - s_j) \quad \text{for } 0 \leq j \leq i \leq n, \\ \|A(s_i, y_i)\| &\leq M \quad \text{for } 0 \leq i \leq n. \end{aligned}$$

Moreover, if $\eta > 0$ and $\|A(s, y) - A(t, x)\| \leq \eta$ for $(s, y) \in \Omega \cap S_r(t, x)$, then

$$\|x + (s_n - t)A(t, x) - y_n\| \leq (\varepsilon + \eta)(s_n - t). \quad (2.3)$$

(ii) Let $\eta > 0$ and $\|A(s, y) - A(t, x)\| \leq \eta$ for $(s, y) \in \Omega \cap S_r(t, x)$. If a sequence $\{(s_i, y_i)\}_{i=0}^\infty$ in Ω satisfies

$$t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma \quad \text{and} \quad \lim_{i \rightarrow \infty} s_i = t + \sigma, \quad (2.4)$$

$$\begin{aligned} \|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| &\leq \varepsilon(s_i - s_{i-1}) \\ \text{for } i \geq 1, \text{ where } y_0 &= x, \end{aligned} \quad (2.5)$$

then $\hat{y} = \lim_{i \rightarrow \infty} y_i$ exists in X , $\hat{y} \in \Omega(t + \sigma)$ and

$$\|x + \sigma A(t, x) - \hat{y}\| \leq (\varepsilon + \eta)\sigma. \quad (2.6)$$

PROOF. To prove (i), let $\{(s_i, y_i)\}_{i=0}^n$ be a sequence in Ω satisfying (2.1) and (2.2). We first show inductively that $(s_i, y_i) \in S_r(t, x)$ for $0 \leq i \leq n$. It is obvious that $(s_0, y_0) \in S_r(t, x)$. Let k be a nonnegative integer such that $k < n$ and assume that $(s_i, y_i) \in S_r(t, x)$ for $0 \leq i \leq k$. From (2.2) we obtain

$$\|y_{i-1} - y_i\| \leq (s_i - s_{i-1})\|A(s_{i-1}, y_{i-1})\| + \varepsilon(s_i - s_{i-1})$$

for $1 \leq i \leq n$. Since $\|A(s_i, y_i)\| \leq M$ for $0 \leq i \leq k$ by assumption, we have

$$\|y_i - y_{i-1}\| \leq (M + \varepsilon)(s_i - s_{i-1})$$

for $1 \leq i \leq k + 1$. Summing up this inequality from $i = 1$ to $i = k + 1$, we find that

$$\|y_{k+1} - x\| \leq (M + \varepsilon)(s_{k+1} - t) < (M + 1)\sigma \leq r.$$

It is obvious that $s_{k+1} - t \leq \sigma < \sigma(M + 1) \leq r$. These mean that $(s_{k+1}, y_{k+1}) \in S_r(t, x)$. Thus, we inductively prove that $(s_i, y_i) \in S_r(t, x)$ for $0 \leq i \leq n$.

Since $(s_k, y_k) \in S_r(t, x)$ for $0 \leq k \leq n$, we have $\|A(s_k, y_k)\| \leq M$ for $0 \leq k \leq n$ and $\|y_k - y_{k-1}\| \leq (M + \varepsilon)(s_k - s_{k-1})$ for $1 \leq k \leq n$. Therefore, we find that

$$\|y_i - y_j\| \leq (M + \varepsilon)(s_i - s_j)$$

for $0 \leq j \leq i \leq n$. To prove (2.3), let $\eta > 0$ and assume that $\|A(s, y) - A(t, x)\| \leq \eta$ for $(s, y) \in \Omega \cap S_r(t, x)$. Since $\{(s_i, y_i); 0 \leq i \leq n\} \subset \Omega \cap S_r(t, x)$, we have $\|A(s_i, y_i) - A(t, x)\| \leq \eta$ for $0 \leq i \leq n$. From (2.2) we see that

$$\begin{aligned} & \|y_{i-1} + (s_i - s_{i-1})A(t, x) - y_i\| \\ & \leq \|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \\ & \quad + \|(s_i - s_{i-1})(A(t, x) - A(s_{i-1}, y_{i-1}))\| \\ & \leq \varepsilon(s_i - s_{i-1}) + \eta(s_i - s_{i-1}) = (\varepsilon + \eta)(s_i - s_{i-1}) \end{aligned}$$

for $1 \leq i \leq n$. Hence

$$\begin{aligned} \|x + (s_n - t)A(t, x) - y_n\| & \leq \sum_{i=1}^n \|y_{i-1} + (s_i - s_{i-1})A(t, x) - y_i\| \\ & \leq (\varepsilon + \eta)(s_n - t). \end{aligned}$$

To prove (ii), let $\{(s_i, y_i)\}_{i=0}^\infty$ be a sequence in Ω satisfying (2.4) and (2.5). From (i) we obtain $\|y_i - y_j\| \leq (M + \varepsilon)(s_i - s_j)$ for $0 \leq j \leq i$. This implies that $\hat{y} = \lim_{i \rightarrow \infty} y_i$ exists in X and is in $\Omega(t + \sigma)$ by (Q2). By (i) again, we note that the inequality (2.3) holds for $n \geq 0$. Passing to the limit in (2.3) as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \|x + \sigma A(t, x) - \hat{y}\| & = \lim_{n \rightarrow \infty} \|x + (s_n - t)A(t, x) - y_n\| \\ & \leq \lim_{n \rightarrow \infty} (\varepsilon + \eta)(s_n - t) = (\varepsilon + \eta)\sigma, \end{aligned}$$

namely, the desired inequality (2.6) is proved. $\square$

The local existence of approximation solutions to (IVP; τ, z) is given by the following proposition, which is essentially shown in [7] and [4]. We give the proof for completeness.

PROPOSITION 2. *Let $(t, x) \in \Omega$ and $\varepsilon \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(t, x)$. Let $\sigma \in (0, r/(M + 1)]$. Then there exists a sequence $\{(s_i, y_i)\}_{i=0}^\infty$ in Ω such that*

- (i) $t = s_0 < s_1 < \cdots < s_i < \cdots < t + \sigma$ and $\lim_{i \rightarrow \infty} s_i = t + \sigma$;
- (ii) $s_i - s_{i-1} \leq \varepsilon$ for $i \geq 1$;

- (iii) $\|y_{i-1} + (s_i - s_{i-1})A(s_{i-1}, y_{i-1}) - y_i\| \leq \varepsilon(s_i - s_{i-1})/2$ for $i \geq 1$, where $y_0 = x$;
 (iv) if $(s, y) \in \Omega \cap S_{(M+1)(s_i - s_{i-1})}(s_{i-1}, y_{i-1})$, then

$$\|A(s, y) - A(s_{i-1}, y_{i-1})\| \leq \varepsilon/4 \quad \text{for } i \geq 1.$$

PROOF. Set $(s_0, y_0) = (t, x)$. Let k be a positive integer and assume that there exists a sequence $\{(s_i, y_i)\}_{i=0}^{k-1}$ in Ω which satisfies the first half of (i) and (ii)–(iv) for $1 \leq i \leq k-1$. We consider a nonnegative number $\hat{h}_k$ defined by the supremum of $h \in [0, \varepsilon]$ such that $h < t + \sigma - s_{k-1}$ and

$$\|A(s, y) - A(s_{k-1}, y_{k-1})\| \leq \varepsilon/4 \quad \text{for } (s, y) \in \Omega \cap S_{\hat{h}(M+1)}(s_{k-1}, y_{k-1}).$$

By the continuity of A , we have $\hat{h}_k > 0$. Thus there exists a number $h_k \in (0, \varepsilon]$ such that $\hat{h}_k/2 < h_k < t + \sigma - s_{k-1}$ and

$$\|A(s, y) - A(s_{k-1}, y_{k-1})\| \leq \varepsilon/4 \quad \text{for } (s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1}), \quad (2.7)$$

where $r_k = h_k(M+1)$. Set $s_k = s_{k-1} + h_k$. Then $s_{k-1} < s_k < t + \sigma$ and conditions (ii) and (iv) with $i = k$ are satisfied. By Lemma 3, $\|A(s_i, y_i)\| \leq M$ for $0 \leq i \leq k-1$. The inequality (2.7) implies that $\|A(s, y)\| \leq M + \varepsilon/4$ for $(s, y) \in \Omega \cap S_{r_k}(s_{k-1}, y_{k-1})$. Hence, Lemma 1, with (t, x) , r , M and η replaced by (s_{k-1}, y_{k-1}) , r_k , $M + \varepsilon/4$ and $\varepsilon/4$ respectively, implies that

$$d(y_{k-1} + h_k A(s_{k-1}, y_{k-1}), \Omega(s_k)) \leq \varepsilon h_k/4.$$

Thus there exists an element $y_k \in \Omega(s_k)$ satisfying (iii) with $i = k$.

We shall show that $\lim_{i \rightarrow \infty} s_i = t + \sigma$. Assume to the contrary that $\hat{s} = \lim_{i \rightarrow \infty} s_i < t + \sigma$. By Lemma 3 (i) we obtain $\|y_i - y_j\| \leq (M + \varepsilon/2)(s_i - s_j)$ for $0 \leq j \leq i$. Hence, $\lim_{i \rightarrow \infty} y_i$ exists in X , and we denote its limit by $\hat{y}$. Since $(\hat{s}, \hat{y}) = \lim_{i \rightarrow \infty} (s_i, y_i)$ in $\mathbf{R} \times X$ and $(s_i, y_i) \in \Omega$ for $i \geq 1$, we have $(\hat{s}, \hat{y}) \in \Omega$ by (Ω2). The continuity of A enables us to choose $\eta \in (0, \varepsilon]$ such that

$$\eta \leq t + \sigma - \hat{s} \quad \text{and} \quad \|A(s, y) - A(\hat{s}, \hat{y})\| \leq \varepsilon/8 \quad \text{for } (s, y) \in \Omega \cap S_{\hat{r}}(\hat{s}, \hat{y}),$$

where $\hat{r} = 2(M+1)\eta$. Choose an integer $i_0 \geq 1$ so that $\hat{s} - s_{i-1} \leq \eta$ and $\|\hat{y} - y_{i-1}\| \leq (M+1)\eta$ for $i \geq i_0$. Then, for $i \geq i_0$ and $(s, y) \in S_{(M+1)\eta}(s_{i-1}, y_{i-1})$, we have

$$|s - \hat{s}| \leq |s - s_{i-1}| + |s_{i-1} - \hat{s}| < (M+1)\eta + \eta \leq 2(M+1)\eta,$$

$$\|y - \hat{y}\| \leq \|y - y_{i-1}\| + \|y_{i-1} - \hat{y}\| < 2(M+1)\eta.$$

Hence $S_{(M+1)\eta}(s_{i-1}, y_{i-1}) \subset S_{\hat{r}}(\hat{s}, \hat{y})$ for $i \geq i_0$. By the choice of η , we see that if $i \geq i_0$, then

$$\begin{aligned} \|A(s, y) - A(s_{i-1}, y_{i-1})\| &\leq \|A(s, y) - A(\hat{s}, \hat{y})\| + \|A(\hat{s}, \hat{y}) - A(s_{i-1}, y_{i-1})\| \\ &\leq \varepsilon/8 + \varepsilon/8 = \varepsilon/4 \end{aligned}$$

for $(s, y) \in \Omega \cap S_{(M+1)\eta}(s_{i-1}, y_{i-1})$. Since $\eta < t + \sigma - s_{i-1}$ for $i \geq 1$, the definition of $\hat{h}_i$ implies that $\eta \leq \hat{h}_i < 2h_i = 2(s_i - s_{i-1})$ for $i \geq i_0$ and the right-hand side tends to zero as $i \rightarrow \infty$. This contradicts the fact that η is positive. $\square$

In what follows, we write $\bar{\omega}([\hat{a}, \hat{b}]) = \sup_{s \in [\hat{a}, \hat{b}]} \omega(s)$ for $[\hat{a}, \hat{b}] \subset [a, b]$. To prove the convergence of the approximate solutions, we need the following Propositions, which are the refinements of the results in [11], [10], [6] and [8].

PROPOSITION 3. *Let $t \in [a, b]$, $(x, \hat{x}) \in \Omega(t) \times \Omega(t)$ and $\eta, \hat{\eta} \in (0, 1)$. Let $r > 0$ and $M > 0$ be numbers such that $t + r < b$,*

$$\|A(s, z)\| \leq M \quad \text{and} \quad \|A(s, z) - A(t, x)\| \leq \eta/4 \quad \text{for } (s, z) \in \Omega \cap S_r(t, x),$$

$$\|A(s, \hat{z})\| \leq M \quad \text{and} \quad \|A(s, \hat{z}) - A(t, \hat{x})\| \leq \hat{\eta}/4 \quad \text{for } (s, \hat{z}) \in \Omega \cap S_r(t, \hat{x}).$$

Let $\sigma \in (0, r/(M+1))$. Then there exists a pair $(y, \hat{y}) \in \Omega(t+\sigma) \times \Omega(t+\sigma)$ such that

$$\|x + \sigma A(t, x) - y\| \leq \eta\sigma, \quad (2.8)$$

$$\|\hat{x} + \sigma A(t, \hat{x}) - \hat{y}\| \leq \hat{\eta}\sigma, \quad (2.9)$$

$$V(t+\sigma, y, \hat{y}) \leq \exp(\sigma \bar{\omega}([t, t+\sigma]))(V(t, x, \hat{x}) + L(\eta + \hat{\eta})\sigma). \quad (2.10)$$

PROOF. We shall show that there exist two sequences $\{(s_j, z_j)\}_{j=0}^\infty$ and $\{(s_j, \hat{z}_j)\}_{j=0}^\infty$ in Ω such that

$$t = s_0 < s_1 < \cdots < s_j < \cdots < t + \sigma \quad \text{and} \quad \lim_{j \rightarrow \infty} s_j = t + \sigma, \quad (2.11)$$

$$\begin{aligned} \|z_{j-1} + (s_j - s_{j-1})A(s_{j-1}, z_{j-1}) - z_j\| &\leq 3\eta(s_j - s_{j-1})/4 \\ \text{for } j \geq 1, \text{ where } z_0 &= x, \end{aligned} \quad (2.12)$$

$$\begin{aligned} \|\hat{z}_{j-1} + (s_j - s_{j-1})A(s_{j-1}, \hat{z}_{j-1}) - \hat{z}_j\| &\leq 3\hat{\eta}(s_j - s_{j-1})/4 \\ \text{for } j \geq 1, \text{ where } \hat{z}_0 &= \hat{x}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} &(V(s_j, z_j, \hat{z}_j) - V(s_{j-1}, z_{j-1}, \hat{z}_{j-1})) / (s_j - s_{j-1}) \\ &\leq \omega(s_{j-1})V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + L(\eta + \hat{\eta}) \quad \text{for } j \geq 1. \end{aligned} \quad (2.14)$$

Set $(s_0, z_0, \hat{z}_0) = (t, x, \hat{x})$ and assume that sequences $\{(s_j, z_j)\}_{j=0}^{i-1}$ and $\{(s_j, \hat{z}_j)\}_{j=0}^{i-1}$ in Ω with $i \geq 1$ satisfy the first half of (2.11) and (2.12)–(2.14) for $1 \leq j \leq i-1$. Then we need to show that there exist $s_i \in \mathbf{R}$, $z_i \in \Omega(s_i)$ and $\hat{z}_i \in \Omega(s_i)$

such that $s_{i-1} < s_i < t + \sigma$ and (2.12)–(2.14) with $j = i$ are satisfied. Let $\hat{h}_i$ denote the supremum of all $h \geq 0$ such that $h < t + \sigma - s_{i-1}$ and

$$\begin{aligned} & V(s_{i-1} + h, z_{i-1} + hA(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1}, \hat{z}_{i-1})) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) \\ & \leq h(\omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4). \end{aligned}$$

Since $\hat{h}_i > 0$ by (Ω4), there exists a number $h_i > 0$ such that $\hat{h}_i/2 < h_i < t + \sigma - s_{i-1}$ and

$$\begin{aligned} & V(s_{i-1} + h, z_{i-1} + hA(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + hA(s_{i-1}, \hat{z}_{i-1})) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) \\ & \leq h(\omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4). \end{aligned} \quad (2.15)$$

Set $s_i = s_{i-1} + h_i$. It is obvious that $s_{i-1} < s_i < t + \sigma$. To prove that $S_{(M+1)h_i}(s_{i-1}, z_{i-1}) \subset S_r(t, x)$, we note by Lemma 3 (i) with $\varepsilon = 3\eta/4$ that

$$\|z_{i-1} - x\| \leq (M + 3\eta/4)(s_{i-1} - t) < (M + 1)(s_{i-1} - t).$$

If $(s, z) \in S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$\begin{aligned} |s - t| & \leq |s - s_{i-1}| + |s_{i-1} - t| < (M + 1)(h_i + s_{i-1} - t) \\ & = (M + 1)(s_i - t) \leq (M + 1)\sigma \leq r \end{aligned}$$

and

$$\|z - x\| \leq \|z - z_{i-1}\| + \|z_{i-1} - x\| < (M + 1)(h_i + s_{i-1} - t) \leq r.$$

This means that $S_{(M+1)h_i}(s_{i-1}, z_{i-1}) \subset S_r(t, x)$. By assumption, we have

$$\|A(s, z)\| \leq M \quad \text{and} \quad \|A(s, z) - A(t, x)\| \leq \eta/4 \quad (2.16)$$

for $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$. From the second inequality of (2.16), we see that if $(s, z) \in \Omega \cap S_{(M+1)h_i}(s_{i-1}, z_{i-1})$, then

$$\begin{aligned} \|A(s, z) - A(s_{i-1}, z_{i-1})\| & \leq \|A(s, z) - A(t, x)\| + \|A(s_{i-1}, z_{i-1}) - A(t, x)\| \\ & \leq \eta/4 + \eta/4 = \eta/2. \end{aligned}$$

Hence, by Lemma 1 with $r = (M + 1)h_i$, $(t, x) = (s_{i-1}, z_{i-1})$ and $h = h_i$, we find that

$$d(z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \Omega(s_i)) \leq h_i \eta/2 = \eta(s_i - s_{i-1})/2.$$

This implies that there exists $z_i \in \Omega(s_i)$ such that (2.12) holds true for $j = i$. Similarly, we can show that there exists $\hat{z}_i \in \Omega(s_i)$ satisfying (2.13) with $j = i$.

By (V1) we obtain (2.14) with $j = i$ by the inequality (2.15) combined with (2.12) and (2.13) with $j = i$. Indeed, we have

$$\begin{aligned}
& (V(s_i, z_i, \hat{z}_i) - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}))/h_i \\
&= (V(s_i, z_i, \hat{z}_i) - V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))/h_i \\
&\quad + (V(s_i, z_{i-1} + h_i A(s_{i-1}, z_{i-1}), \hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1})) \\
&\quad - V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}))/h_i \\
&\leq L(\|z_i - (z_{i-1} + h_i A(s_{i-1}, z_{i-1}))\| + \|\hat{z}_i - (\hat{z}_{i-1} + h_i A(s_{i-1}, \hat{z}_{i-1}))\|)/h_i \\
&\quad + \omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4 \\
&\leq 3(\eta + \hat{\eta})L/4 + \omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + (\eta + \hat{\eta})L/4 \\
&\leq \omega(s_{i-1})V(s_{i-1}, z_{i-1}, \hat{z}_{i-1}) + L(\eta + \hat{\eta}).
\end{aligned}$$

It remains to prove the second half of (2.11). Assume to the contrary that $s_\infty = \lim_{j \rightarrow \infty} s_j < t + \sigma$. Lemma 3 (i) asserts that $\{z_j\}$ and $\{\hat{z}_j\}$ are Cauchy sequences in X , since

$$\begin{aligned}
\limsup_{i,j \rightarrow \infty} \|z_i - z_j\| &\leq \limsup_{i,j \rightarrow \infty} (M + 3\eta/4)(s_i - s_j) = 0, \\
\limsup_{i,j \rightarrow \infty} \|\hat{z}_i - \hat{z}_j\| &\leq \limsup_{i,j \rightarrow \infty} (M + 3\hat{\eta}/4)(s_i - s_j) = 0.
\end{aligned}$$

This implies that $z_\infty = \lim_{j \rightarrow \infty} z_j$ and $\hat{z}_\infty = \lim_{j \rightarrow \infty} \hat{z}_j$ exist in X and are in $\Omega(s_\infty)$ by (Ω2). By (Ω4), we choose a number $h > 0$ so that $h < t + \sigma - s_\infty$ and

$$\begin{aligned}
& \{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\
&\leq \omega(s_\infty)V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/8.
\end{aligned} \tag{2.17}$$

Let $r_j = s_\infty + h - s_{j-1}$ for $j \geq 1$. Then we have $r_j < t + \sigma - s_{j-1}$ for $j \geq 1$ and $r_j \rightarrow h$ as $j \rightarrow \infty$. Since $\hat{h}_j < 2h_j = 2(s_j - s_{j-1}) \rightarrow 0$ as $j \rightarrow \infty$, there exists an integer $j_0 \geq 1$ such that $\hat{h}_j < r_j$ for $j \geq j_0$. By the definition of $\hat{h}_j$, we have

$$\begin{aligned}
& \{V(s_{j-1} + r_j, z_{j-1} + r_j A(s_{j-1}, z_{j-1}), \hat{z}_{j-1} + r_j A(s_{j-1}, \hat{z}_{j-1})) - V(s_{j-1}, z_{j-1}, \hat{z}_{j-1})\}/r_j \\
&> \omega(s_{j-1})V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + (\eta + \hat{\eta})L/4
\end{aligned}$$

for $j \geq j_0$. Since $s_{j-1} \rightarrow s_\infty$, $z_{j-1} \rightarrow z_\infty$, $\hat{z}_{j-1} \rightarrow \hat{z}_\infty$ and $r_j \rightarrow h$ as $j \rightarrow \infty$ and $s_{j-1} + r_j = s_\infty + h$ for $j \geq 1$, from (V1) and (V3) we obtain

$$\begin{aligned}
& \{V(s_\infty + h, z_\infty + hA(s_\infty, z_\infty), \hat{z}_\infty + hA(s_\infty, \hat{z}_\infty)) - V(s_\infty, z_\infty, \hat{z}_\infty)\}/h \\
&\geq \omega(s_\infty)V(s_\infty, z_\infty, \hat{z}_\infty) + (\eta + \hat{\eta})L/4,
\end{aligned}$$

which contradicts to (2.17).

We now turn to the proof of the existence of pair $(y, \hat{y}) \in \Omega(t) \times \Omega(t)$ satisfying (2.8)–(2.10). We apply Lemma 3 (ii) to show that $y = \lim_{j \rightarrow \infty} z_j$ and $\hat{y} = \lim_{j \rightarrow \infty} \hat{z}_j$ exist in X and are in $\Omega(t + \sigma)$ and that they satisfy (2.8) and (2.9), that is,

$$\|x + \sigma A(t, x) - y\| \leq (3\eta/4 + \eta/4)\sigma \leq \eta\sigma,$$

$$\|\hat{x} + \sigma A(t, \hat{x}) - \hat{y}\| \leq (3\hat{\eta}/4 + \hat{\eta}/4)\sigma \leq \hat{\eta}\sigma.$$

We note here that $1 + t \leq e^t$ for $t \geq 0$. We deduce from (2.14) that

$$V(s_j, z_j, \hat{z}_j) \leq \exp(h_j \bar{\omega}([t, t + \sigma]))(V(s_{j-1}, z_{j-1}, \hat{z}_{j-1}) + h_j L(\eta + \hat{\eta}))$$

for $j \geq 1$. Hence, we inductively show that

$$V(s_j, z_j, \hat{z}_j) \leq \exp((s_j - t) \bar{\omega}([t, t + \sigma]))(V(t, x, \hat{x}) + L(\eta + \hat{\eta})(s_j - t))$$

for $j \geq 0$. Thus we obtain (2.10) by letting $j \rightarrow \infty$. $\square$

PROPOSITION 4. *Let $(\tau, z) \in \Omega$ and $\lambda, \mu \in (0, 1/2)$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M + 1))$. For each $\varepsilon \in \{\lambda, \mu\}$, let $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ be a sequence in Ω satisfying the following conditions:*

- (i) $\tau = t_0^\varepsilon < t_1^\varepsilon < \cdots < t_i^\varepsilon < \cdots < \tau + \sigma$ and $\lim_{i \rightarrow \infty} t_i^\varepsilon = \tau + \sigma$;
- (ii) $t_i^\varepsilon - t_{i-1}^\varepsilon \leq \varepsilon$ for $i \geq 1$;
- (iii) $\|x_{i-1}^\varepsilon + (t_i^\varepsilon - t_{i-1}^\varepsilon)A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon) - x_i^\varepsilon\| \leq \varepsilon(t_i^\varepsilon - t_{i-1}^\varepsilon)/2$ for $i \geq 1$, where $x_0^\varepsilon = z$;
- (iv) if $(s, y) \in \Omega \cap S_{(M+1)(t_i^\varepsilon - t_{i-1}^\varepsilon)}(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)$, then

$$\|A(s, y) - A(t_{i-1}^\varepsilon, x_{i-1}^\varepsilon)\| \leq \varepsilon/4 \quad \text{for } i \geq 1.$$

Let $\{s_k\}_{k=0}^\infty$ be a sequence such that $s_k < s_{k+1}$ for $k \geq 0$ and

$$\{s_k; k = 0, 1, 2, \dots\} = \{t_i^\lambda; i = 0, 1, 2, \dots\} \cup \{t_j^\mu; j = 0, 1, 2, \dots\}.$$

Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}_{k=0}^\infty$ in $X \times X$ such that $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ for each $k \geq 0$ and the following three properties are satisfied:

- (a) if $s_k = t_i^\lambda$, then $z_k^\lambda = x_i^\lambda$; if $s_k = t_j^\mu$, then $z_k^\mu = x_j^\mu$;
- (b) for each $\varepsilon = \lambda, \mu$, we have

$$\begin{aligned} & \sum_{j=q}^k \|z_{j-1}^\varepsilon + (s_j - s_{j-1})A(s_{j-1}, z_{j-1}^\varepsilon) - z_j^\varepsilon\| \\ & \leq 2\varepsilon(s_k - s_{q-1}) + 3\varepsilon \sum_{t_i^\varepsilon \in \{s_q, \dots, s_k\}} (t_i^\varepsilon - t_{i-1}^\varepsilon) \end{aligned}$$

for $1 \leq q \leq k$ and $k \geq 1$;

(c) for $k \geq 0$,

$$V(s_k, z_k^\lambda, z_k^\mu) \leq \exp((s_k - \tau)\bar{\omega}([\tau, s_k]))\{2L(\lambda + \mu)(s_k - \tau) + \eta_k(\lambda, \mu)\},$$

where

$$\eta_k(\lambda, \mu) = 3L \left(\lambda \sum_{t_i^\lambda \in \{s_1, \dots, s_k\}} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu \in \{s_1, \dots, s_k\}} (t_j^\mu - t_{j-1}^\mu) \right).$$

PROOF. Set $z_0^\varepsilon = z$ for each $\varepsilon = \lambda, \mu$. Assume that sequences $\{(s_k, z_k^\lambda)\}_{k=0}^{l-1}$ and $\{(s_k, z_k^\mu)\}_{k=0}^{l-1}$ in Ω with $l \geq 1$ satisfy properties (a)–(c) for $0 \leq k \leq l-1$. Let i and j be positive integers such that $t_{i-1}^\lambda < s_l \leq t_i^\lambda$ and $t_{j-1}^\mu < s_l \leq t_j^\mu$, respectively. By Lemma 3 (i) with $\varepsilon = \lambda/2$ we obtain $\|x_{i-1}^\lambda - z\| \leq (M + \lambda/2)(t_{i-1}^\lambda - \tau)$. If $(s, y) \in S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda)$, then we get

$$\begin{aligned} |s - \tau| &\leq |s - t_{i-1}^\lambda| + |t_{i-1}^\lambda - \tau| < (M + 1)(t_i^\lambda - t_{i-1}^\lambda) + (t_{i-1}^\lambda - \tau) \\ &\leq (M + 1)\sigma \leq R \end{aligned}$$

and

$$\begin{aligned} \|y - z\| &\leq \|y - x_{i-1}^\lambda\| + \|x_{i-1}^\lambda - z\| \\ &< (M + 1)(t_i^\lambda - t_{i-1}^\lambda) + (M + \lambda/2)(t_{i-1}^\lambda - \tau) < (M + 1)\sigma \leq R. \end{aligned}$$

Hence $S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda) \subset S_R(\tau, z)$. This implies that

$$\|A(s, y)\| \leq M \quad \text{for } (s, y) \in \Omega \cap S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda). \quad (2.18)$$

We shall show that for each $\varepsilon = \lambda, \mu$,

$$\|A(s, y)\| \leq M \quad \text{and} \quad \|A(s, y) - A(s_{l-1}, z_{l-1}^\varepsilon)\| \leq \varepsilon/2 \quad (2.19)$$

for $(s, y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^\varepsilon)$. By the definition of $\{s_k\}$ we observe that

$$\begin{aligned} t_{i-1}^\lambda &\leq s_{l-1} < s_l \leq t_i^\lambda, & t_{j-1}^\mu &\leq s_{l-1} < s_l \leq t_j^\mu, \\ t_{i-1}^\lambda &= s_p \quad \text{for some } 0 \leq p \leq l-1, & \text{and} \\ t_{j-1}^\mu &= s_q \quad \text{for some } 0 \leq q \leq l-1. \end{aligned}$$

By the hypothesis (a) of induction, we have $z_p^\lambda = x_{i-1}^\lambda$ and $z_q^\mu = x_{j-1}^\mu$. If $0 \leq p < l-1$, then the set $\{s_{p+1}, \dots, s_{l-1}\}$ contains no points t_i^λ . By the hypothesis (b) of induction, we have

$$\|z_{k-1}^\lambda + (s_k - s_{k-1})A(s_{k-1}, z_{k-1}^\lambda) - z_k^\lambda\| \leq 2\lambda(s_k - s_{k-1}) \quad (2.20)$$

for $k = p + 1, \dots, l - 1$. By (2.18) and (2.20), we use Lemma 3 (i) with $(t, x) = (t_{i-1}^\lambda, x_{i-1}^\lambda) = (s_p, z_p^\lambda)$, $\varepsilon = 2\lambda$ and $r = (M + 1)(t_i^\lambda - t_{i-1}^\lambda)$ to obtain $\|z_{i-1}^\lambda - z_p^\lambda\| \leq (M + 2\lambda)(s_{l-1} - s_p)$. This is valid for $p = l - 1$. If $(s, y) \in S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^\lambda)$, then we get

$$\begin{aligned} |s - t_{i-1}^\lambda| &\leq |s - s_{l-1}| + |s_{l-1} - t_{i-1}^\lambda| \\ &< (M + 1)(s_l - s_{l-1}) + (s_{l-1} - t_{i-1}^\lambda) \leq (M + 1)(t_i^\lambda - t_{i-1}^\lambda), \\ \|y - x_{i-1}^\lambda\| &\leq \|y - z_{l-1}^\lambda\| + \|z_{l-1}^\lambda - x_{i-1}^\lambda\| \\ &< (M + 1)(s_l - s_{l-1}) + (M + 2\lambda)(s_{l-1} - s_p) \leq (M + 1)(t_i^\lambda - t_{i-1}^\lambda). \end{aligned}$$

This means that

$$S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^\lambda) \subset S_{(M+1)(t_i^\lambda - t_{i-1}^\lambda)}(t_{i-1}^\lambda, x_{i-1}^\lambda). \quad (2.21)$$

Thus, the claim (2.19) with $\varepsilon = \lambda$ follows from (2.18) and condition (iv). Indeed,

$$\begin{aligned} \|A(s, y) - A(s_{l-1}, z_{l-1}^\lambda)\| &\leq \|A(s, y) - A(t_{i-1}^\lambda, x_{i-1}^\lambda)\| + \|A(t_{i-1}^\lambda, x_{i-1}^\lambda) - A(s_{l-1}, z_{l-1}^\lambda)\| \\ &\leq \lambda/4 + \lambda/4 = \lambda/2 \end{aligned}$$

for $(s, y) \in \Omega \cap S_{(M+1)(s_l - s_{l-1})}(s_{l-1}, z_{l-1}^\lambda)$. We apply the above argument again, with p and i replaced by q and j , to show that (2.19) holds true for $\varepsilon = \mu$.

By virtue of (2.19), we deduce from Proposition 3 with $t = s_{l-1}$, $(x, \hat{x}) = (z_{l-1}^\lambda, z_{l-1}^\mu)$, $\eta = 2\lambda$, $\hat{\eta} = 2\mu$ and $r = (M + 1)(s_l - s_{l-1})$ that there exists a pair $(y_l^\lambda, y_l^\mu) \in \Omega(s_{l-1} + (s_l - s_{l-1})) \times \Omega(s_{l-1} + (s_l - s_{l-1})) = \Omega(s_l) \times \Omega(s_l)$ satisfying

$$\|z_{l-1}^\varepsilon + (s_l - s_{l-1})A(s_{l-1}, z_{l-1}^\varepsilon) - y_l^\varepsilon\| \leq 2\varepsilon(s_l - s_{l-1}) \quad \text{for } \varepsilon = \lambda, \mu, \quad (2.22)$$

$$\begin{aligned} V(s_l, y_l^\lambda, y_l^\mu) &\leq \exp((s_l - s_{l-1})\bar{\omega}([s_{l-1}, s_l])) \\ &\quad \times (V(s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) + 2L(\lambda + \mu)(s_l - s_{l-1})). \end{aligned} \quad (2.23)$$

We define $(z_l^\lambda, z_l^\mu) \in \Omega(s_l) \times \Omega(s_l)$ by

$$z_l^\lambda = \begin{cases} y_l^\lambda & \text{for } s_l < t_i^\lambda, \\ x_i^\lambda & \text{for } s_l = t_i^\lambda \end{cases} \quad \text{and} \quad z_l^\mu = \begin{cases} y_l^\mu & \text{for } s_l < t_j^\mu, \\ x_j^\mu & \text{for } s_l = t_j^\mu. \end{cases}$$

If $s_l = t_i^\lambda$, then by condition (iii) we have

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - z_l^\lambda\| \leq (s_l - t_{i-1}^\lambda)\lambda/2,$$

while in view of (2.18) and (iv) we find, by applying Lemma 3 (i), with $\varepsilon = 2\lambda$, $\eta = \lambda/4$, $r = (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $(t, x) = (t_{i-1}^\lambda, x_{i-1}^\lambda)$, to (2.20) and (2.22), that

$$\|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \leq (2\lambda + \lambda/4)(s_l - t_{i-1}^\lambda).$$

These inequalities together yield

$$\begin{aligned} \|z_l^\lambda - y_l^\lambda\| &\leq \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - y_l^\lambda\| \\ &\quad + \|x_{i-1}^\lambda + (s_l - t_{i-1}^\lambda)A(t_{i-1}^\lambda, x_{i-1}^\lambda) - z_l^\lambda\| \\ &\leq (9/4 + 1/2)\lambda(s_l - t_{i-1}^\lambda) \leq 3\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda). \end{aligned} \quad (2.24)$$

Similarly, we get

$$\|z_l^\mu - y_l^\mu\| \leq 3\mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu). \quad (2.25)$$

Combining (2.24) and (2.25) with (2.22), and adding the resulting inequality to the inequality (b) with $k = l - 1$, we conclude that the desired property (b) holds true for $k = l$.

Finally, we show that (c) is true for $k = l$. Using (2.24), (2.25) and (V1) we have

$$\begin{aligned} |V(s_l, z_l^\lambda, z_l^\mu) - V(s_l, y_l^\lambda, y_l^\mu)| &\leq L(\|z_l^\lambda - y_l^\lambda\| + \|z_l^\mu - y_l^\mu\|) \\ &\leq 3L \left(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu) \right). \end{aligned}$$

Combining this and (2.23), we obtain

$$\begin{aligned} V(s_l, z_l^\lambda, z_l^\mu) &\leq V(s_l, y_l^\lambda, y_l^\mu) + 3L \left(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu) \right) \\ &\leq \exp((s_l - s_{l-1})\bar{\omega}([s_{l-1}, s_l]))(V(s_{l-1}, z_{l-1}^\lambda, z_{l-1}^\mu) \\ &\quad + 2L(\lambda + \mu)(s_l - s_{l-1})) \\ &\quad + 3L \left(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu) \right) \end{aligned}$$

$$\begin{aligned}
&\leq \exp((s_l - \tau)\bar{\omega}([\tau, s_l]))(2L(\lambda + \mu)(s_l - \tau) + \eta_{l-1}(\lambda, \mu)) \\
&\quad + 3L \left(\lambda \sum_{t_i^\lambda = s_l} (t_i^\lambda - t_{i-1}^\lambda) + \mu \sum_{t_j^\mu = s_l} (t_j^\mu - t_{j-1}^\mu) \right) \\
&\leq \exp((s_l - \tau)\bar{\omega}([\tau, s_l]))(2L(\lambda + \mu)(s_l - \tau) + \eta_l(\lambda, \mu)).
\end{aligned}$$

This means that (c) is true for $k = l$, and the proof is completed. $\square$

The following is a local existence theorem of solutions to (IVP; τ, z).

THEOREM 3. *Let $(\tau, z) \in \Omega$. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Let $\sigma \in (0, R/(M + 1)]$. Then there exists a solution u to (IVP; τ, z) on $[\tau, \tau + \sigma]$ such that*

$$\|u(t) - u(s)\| \leq M|t - s| \quad \text{for } t, s \in [\tau, \tau + \sigma].$$

PROOF. Let $\varepsilon \in (0, 1/2)$. Then, by Proposition 2, there exists a sequence $\{(t_i^\varepsilon, x_i^\varepsilon)\}_{i=0}^\infty$ in Ω satisfying (i)–(iv) of Proposition 4. Let $u^\varepsilon : [\tau, \tau + \sigma] \rightarrow X$ be the function defined by $u^\varepsilon(t) = x_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i \geq 0$. We want to prove that the family $\{u^\varepsilon\}$ converges in X uniformly on $[\tau, \tau + \sigma]$ as $\varepsilon \downarrow 0$.

Let $\lambda, \mu \in (0, 1/2)$ and let $\{s_k\}_{k=0}^\infty$ be a sequence defined as in Proposition 4. Then there exists a sequence $\{(z_k^\lambda, z_k^\mu)\}$ in $X \times X$ satisfying $(z_k^\lambda, z_k^\mu) \in \Omega(s_k) \times \Omega(s_k)$ for $k \geq 0$ and (a)–(c) of Proposition 4. We first prove that

$$\sup_{k \geq 0} \|z_k^\lambda - z_k^\mu\| \rightarrow 0 \quad \text{as } \lambda, \mu \downarrow 0. \quad (2.26)$$

Assume to the contrary that there exist $\varepsilon_0 > 0$, two null sequences $\{\lambda_n\}$ and $\{\mu_n\}$ of positive numbers, and a sequence $\{k_n\}$ of nonnegative integers such that

$$\|z_{k_n}^{\lambda_n} - z_{k_n}^{\mu_n}\| \geq \varepsilon_0 \quad \text{for } n \geq 1. \quad (2.27)$$

Since the sequence $\{s_{k_n}\}$ is bounded as $n \rightarrow \infty$, it has a convergent subsequence $\{s_{k_{n_l}}\}$. Since $(z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \in \Omega(s_{k_{n_l}}) \times \Omega(s_{k_{n_l}})$ for $l \geq 1$, and since

$$V(s_{k_{n_l}}, z_{k_{n_l}}^{\lambda_{n_l}}, z_{k_{n_l}}^{\mu_{n_l}}) \leq 5L \exp(\sigma \bar{\omega}([\tau, \tau + \sigma]))(\lambda_{n_l} + \mu_{n_l})\sigma \quad \text{for } l \geq 1$$

by Proposition 4 (c), we deduce from condition (V4) that $\lim_{l \rightarrow \infty} \|z_{k_{n_l}}^{\lambda_{n_l}} - z_{k_{n_l}}^{\mu_{n_l}}\| = 0$. This is a contradiction to (2.27).

Let $t \in [\tau, \tau + \sigma]$. Let $k \geq 1$ be an integer such that $t \in [s_{k-1}, s_k)$. Let i and j be positive integers such that $t_{i-1}^\lambda \leq s_{k-1} < s_k \leq t_i^\lambda$ and $t_{j-1}^\mu \leq$

$s_{k-1} < s_k \leq t_j^\mu$, respectively. Then we have, in a similar way to the derivation of (2.21), $\|z_{k-1}^\lambda - x_{i-1}^\lambda\| \leq (M+1)(t_i^\lambda - t_{i-1}^\lambda)$ and $\|z_{k-1}^\mu - x_{j-1}^\mu\| \leq (M+1)(t_j^\mu - t_{j-1}^\mu)$. Since

$$\begin{aligned} \|u^\lambda(t) - u^\mu(t)\| &\leq \|x_{i-1}^\lambda - z_{k-1}^\lambda\| + \|z_{k-1}^\lambda - z_{k-1}^\mu\| + \|z_{k-1}^\mu - x_{j-1}^\mu\| \\ &\leq (M+1)(\lambda + \mu) + \|z_{k-1}^\lambda - z_{k-1}^\mu\|, \end{aligned}$$

we observe from (2.26) that the family $\{u^\varepsilon(t)\}$ is uniformly Cauchy on $[\tau, \tau + \sigma]$. By Lemma 3 (i) we obtain

$$\|u^\varepsilon(t) - u^\varepsilon(s)\| \leq (M + \varepsilon/2)(|t - s| + 2\varepsilon) \quad \text{for } t, s \in [\tau, \tau + \sigma]$$

and $\varepsilon \in (0, 1/2)$. These facts imply that there exists a continuous function u defined on $[\tau, \tau + \sigma]$ such that $\sup_{t \in [\tau, \tau + \sigma]} \|u^\varepsilon(t) - u(t)\| \rightarrow 0$ as $\varepsilon \downarrow 0$. It is clear that $u(\tau) = z$ and $\|u(t) - u(s)\| \leq M|t - s|$ for $t, s \in [\tau, \tau + \sigma]$. Let $\tau^\varepsilon : [\tau, \tau + \sigma] \rightarrow \mathbf{R}$ be the function defined by $\tau^\varepsilon(t) = t_i^\varepsilon$ for $t \in [t_i^\varepsilon, t_{i+1}^\varepsilon)$ and $i \geq 0$. Then $\tau \leq \tau^\varepsilon(t) \leq t < \tau + \sigma$ and $\lim_{\varepsilon \downarrow 0} \tau^\varepsilon(t) = t$ for $t \in [\tau, \tau + \sigma]$. From Proposition 4 (iii) we deduce that

$$\left\| u^\varepsilon(t_i^\varepsilon) - u^\varepsilon(0) - \int_\tau^{t_i^\varepsilon} A(\tau^\varepsilon(s), u^\varepsilon(s)) ds \right\| \leq \varepsilon(t_i^\varepsilon - \tau)/2 \leq \varepsilon\sigma/2 \quad (2.28)$$

for $i \geq 0$. Since $(\tau^\varepsilon(t), u^\varepsilon(t)) \in \Omega$ and $\|A(\tau^\varepsilon(t), u^\varepsilon(t))\| \leq M$ for $t \in [\tau, \tau + \sigma]$ and since $(\tau^\varepsilon(t), u^\varepsilon(t)) \rightarrow (t, u(t))$, we have $(t, u(t)) \in \Omega$ and $A(\tau^\varepsilon(t), u^\varepsilon(t)) \rightarrow A(t, u(t))$ for $t \in [\tau, \tau + \sigma]$ as $\varepsilon \downarrow 0$, by (Ω2) and (Ω1) respectively. From (2.28) we obtain

$$u(t) - u(0) = \int_\tau^t A(s, u(s)) ds$$

for $t \in [\tau, \tau + \sigma]$. Since $t \mapsto A(t, u(t))$ is continuous on $[\tau, \tau + \sigma]$, u is a solution to (IVP; τ, z) on $[\tau, \tau + \sigma]$. Since the uniqueness follows from Proposition 1, the proof is completed. $\square$

3. Global existence of solutions

In this section we investigate the intervals where the solutions to (IVP; τ, z) exist under assumptions (Ω1)–(Ω4). We follow the arguments in [4], [6] and [7].

PROPOSITION 5. *Let $(\tau, z) \in \Omega$. Then there exists $c_0 \in (\tau, b)$ such that for any $c \in (\tau, c_0)$, the following properties are satisfied:*

- (i) (IVP; τ, z) has a solution u on $[\tau, c]$.

- (ii) For any $\varepsilon > 0$, there exists a number $r \in (0, c - \tau)$ which satisfies the following:
- (a) (IVP; t, x) has a solution v on $[t, c]$ for any $(t, x) \in \Omega \cap S_r(\tau, z)$,
 - (b) if $(t, x), (\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$, v and $\hat{v}$ are solutions to (IVP; t, x) on $[t, c]$ and (IVP; $\hat{t}, \hat{x}$) on $[\hat{t}, c]$ respectively, then $V(s, v(s), \hat{v}(s)) < \varepsilon$ for $s \in [t, c] \cap [\hat{t}, c]$.

PROOF. Let $R > 0$ and $M > 0$ be numbers such that $\tau + R < b$ and $\|A(t, x)\| \leq M$ for $(t, x) \in \Omega \cap S_R(\tau, z)$, and set $c_0 = \tau + R/(M + 1)$. We shall show that for any number $c \in (\tau, c_0)$, the desired properties are satisfied. The first property (i) follows from Theorem 3.

We shall show that such a number c has the second property (ii). Let $\varepsilon > 0$. We take $\delta > 0$ so that $\exp(\int_{\tau}^s \omega(\theta) d\theta) \delta < \varepsilon$ for any $s \in [a, c]$. Next, we choose $r > 0$ so small that $\tau + r < c \leq \tau + (R - r)/(M + 1) - r$ and

$$2L(M + 1)r \leq \exp\left(\int_{\tau}^s \omega(\theta) d\theta\right) \delta \quad (3.1)$$

for $s \in [\tau - r, \tau + r] \cap [a, b]$. To prove (a), let $(t, x) \in \Omega \cap S_r(\tau, z)$. Set $\hat{r} = R - r$. Since $\tau + r < c < \tau + R/(M + 1) < \tau + R$, we have $\hat{r} > 0$. Moreover, we have $t + \hat{r} = (t - \tau) + \tau + \hat{r} \leq r + \tau + \hat{r} = \tau + R < b$. For $(s, y) \in S_{\hat{r}}(t, x)$, we have

$$|s - \tau| \leq |s - t| + |t - \tau| < \hat{r} + r = R$$

and

$$\|y - z\| \leq \|y - x\| + \|x - z\| < \hat{r} + r = R.$$

Thus $S_{\hat{r}}(t, x) \subset S_R(\tau, z)$. Since $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_{\hat{r}}(t, x)$ and $t + \hat{r} < b$, (IVP; t, x) has a solution v on $[t, t + \hat{r}/(M + 1)]$ by Theorem 3. Since $t + \hat{r}/(M + 1) > \tau - r + (R - r)/(M + 1) \geq c$, we certainly infer that v is defined on $[t, c]$.

To prove (b), let $\hat{v}$ be a solution to (IVP; $\hat{t}, \hat{x}$) on $[\hat{t}, c]$ with $(\hat{t}, \hat{x}) \in \Omega \cap S_r(\tau, z)$. Assume that $\hat{t} \leq t$ without loss of generality. Then

$$\begin{aligned} \|\hat{v}(t) - v(t)\| &= \|\hat{v}(t) - x\| \leq \|\hat{v}(t) - \hat{x}\| + \|\hat{x} - z\| + \|z - x\| \\ &\leq \|\hat{v}(t) - \hat{v}(\hat{t})\| + 2r \leq M(t - \hat{t}) + 2r \\ &= M((t - \tau) + (\tau - \hat{t})) + 2r \leq 2(M + 1)r. \end{aligned}$$

By Remark 1 and (3.1), we have

$$V(t, v(t), \hat{v}(t)) \leq 2L(M + 1)r \leq \exp\left(\int_{\tau}^t \omega(\theta) d\theta\right) \delta.$$

Thus, by Proposition 1, we obtain

$$V(s, v(s), \hat{v}(s)) \leq \exp\left(\int_t^s \omega(\theta) d\theta\right) V(t, v(t), \hat{v}(t)) \leq \exp\left(\int_\tau^s \omega(\theta) d\theta\right) \delta < \varepsilon$$

for $s \in [t, c]$. $\square$

Let $(\tau, z) \in \Omega$ and let u be a solution to $(\text{IVP}; \tau, z)$ which is noncontinuable to the right. We denote its *final time* by $T(\tau, z)$. It is clear that $\tau < T(\tau, z) \leq b$ and u is a solution to $(\text{IVP}; \tau, z)$ on $[\tau, T(\tau, z))$. Since $(\text{IVP}; \tau, z)$ has a unique solution, $T(\tau, z) \in (\tau, b]$ is well-defined for every $(\tau, z) \in \Omega$. We consider T as a function from the metric space Ω into the extended real line $\mathbf{R} \cup \{\infty\}$ endowed with the usual topology.

PROPOSITION 6. *Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Then there exists a number $r > 0$ with $\tau + r < b$ such that $T(t, x) > d$ for any $(t, x) \in \Omega \cap S_r(\tau, z)$.*

PROOF. Let $(\tau, z) \in \Omega$ and let d be a number such that $\tau < d < T(\tau, z)$. Let u be a solution to $(\text{IVP}; \tau, z)$ on $[\tau, d]$. Since the set $\{(s, u(s)); s \in [\tau, d]\}$ is compact in Ω and A is continuous on Ω , there exists a number $M > 0$ such that $\|A(s, u(s))\| < M$ for $s \in [\tau, d]$.

We first prove that there exists a number $R > 0$ such that $\|A(s, x)\| \leq M$ for any $s \in [\tau, d]$ and $x \in \Omega(s)$ satisfying $V(s, x, u(s)) < R$. Assume to the contrary that for any $n \geq 1$ there exist $s_n \in [\tau, d]$ and $x_n \in \Omega(s_n)$ such that $V(s_n, x_n, u(s_n)) < 1/n$ and $\|A(s_n, x_n)\| > M$. Since the sequence $\{s_n\}$ is bounded, there exists a convergent subsequence $\{s_{n_k}\}$ converging to some number $s \in [\tau, d]$. Since $V(s_{n_k}, x_{n_k}, u(s_{n_k})) \rightarrow 0$ as $k \rightarrow \infty$, we have $\|x_{n_k} - u(s_{n_k})\| \rightarrow 0$ as $k \rightarrow \infty$ by (V4). Since $u(s_{n_k}) \rightarrow u(s)$ as $k \rightarrow \infty$, we have $(s_{n_k}, x_{n_k}) \rightarrow (s, u(s))$ as $k \rightarrow \infty$. Thus, by (Q1), we have $\|A(s, u(s))\| \geq M$. This contradicts to the definition of M .

By Proposition 5, we can choose a number c such that $\tau < c < d$ and properties (i) and (ii) in Proposition 5 are satisfied for (τ, z) . Let $\varepsilon > 0$ be a number such that $\varepsilon \exp(\int_c^s \omega(\theta) d\theta) \leq R$ for $s \in [c, d]$, and then choose $r > 0$ so that $\tau + r < c$ and Proposition 5 (ii) is satisfied for the number ε . Let $(t, x) \in \Omega \cap S_r(\tau, z)$. We want to show that $d < T(t, x)$. To this end, assume to the contrary that $T(t, x) \leq d$ and let v be a noncontinuable solution to $(\text{IVP}; t, x)$. Note by Proposition 5 (ii) that $[t, c] \subset [t, T(t, x))$ and $V(c, v(c), u(c)) < \varepsilon$. By Proposition 1, we have

$$\begin{aligned} V(s, v(s), u(s)) &\leq V(c, v(c), u(c)) \exp\left(\int_c^s \omega(\theta) d\theta\right) \\ &< \varepsilon \exp\left(\int_c^s \omega(\theta) d\theta\right) \leq R \end{aligned}$$

for $s \in [c, T(t, x))$. From the fact proved first, we observe that $\|A(s, v(s))\| \leq M$ for $s \in [c, T(t, x))$. Thus $\|v(t) - v(s)\| \leq M|t - s|$ for $t, s \in [c, T(t, x))$. Therefore, $w = \lim_{s \uparrow T(t, x)} v(s)$ exists in X and $(T(t, x), w) \in \Omega$ by $(\Omega 2)$. In view of Theorem 3, this contradicts the fact that v is noncontinuable to the right of $T(t, x)$. Hence $T(t, x) > d$. $\square$

PROPOSITION 7. *Let $(\tau, z) \in \Omega$ and let $\{(\tau_n, z_n)\}_{n \geq 1}$ be a sequence in Ω converging to (τ, z) as $n \rightarrow \infty$. For $n \geq 1$, let u_n be a noncontinuable solution to $(\text{IVP}; \tau_n, z_n)$, and let u be a noncontinuable solution to $(\text{IVP}; \tau, z)$. Assume that $d \in (\tau, b)$ satisfies $d < T(\tau_n, z_n)$ for $n \geq 1$. Then the following assertions hold:*

- (i) $d < T(\tau, z)$.
- (ii) *For any $\sigma \in (\tau, d)$, the sequence $\{u_n\}$ converges to u uniformly on $[\sigma, d]$ as $n \rightarrow \infty$.*

PROOF. Let $c \in (\tau, d)$ be a number with the properties (i) and (ii) in Proposition 5, and let $\tau < \sigma < c$. We may assume that $\tau_n < \sigma < c < d < T(\tau_n, z_n)$ for $n \geq 1$, because $\lim_{n \rightarrow \infty} \tau_n = \tau < d$. Let $\varepsilon > 0$. Let $r \in (0, c - \tau)$ be a number with the property (ii) in Proposition 5 for the number ε . Since $(\tau_n, z_n) \rightarrow (\tau, z)$ as $n \rightarrow \infty$, there exists an integer $n_0 \geq 1$ such that $(\tau_n, z_n) \in \Omega \cap S_r(\tau, z)$ for $n \geq n_0$. By Proposition 5 (ii-b) we observe that if $n, m \geq n_0$, then $V(s, u_m(s), u_n(s)) \leq \varepsilon$ for $s \in [\sigma, c]$ and

$$\begin{aligned} V(t, u_m(t), u_n(t)) &\leq \exp\left(\int_c^t \omega(\theta) d\theta\right) V(c, u_m(c), u_n(c)) \\ &\leq \varepsilon \exp((d - c)\bar{\omega}([c, d])) \end{aligned}$$

for $t \in [c, d]$. By (V4), the sequence $\{u_n\}$ is uniformly Cauchy on $[\sigma, d]$. Define $\hat{u}(t) = \lim_{n \rightarrow \infty} u_n(t)$ for $t \in [\sigma, d]$. Then we observe that $\hat{u}'(t) = A(t, \hat{u}(t))$ for $t \in [\sigma, d]$. By Proposition 5, we observe that if $n \geq n_0$, then $V(s, u_n(s), u(s)) \leq \varepsilon$ for $s \in [\sigma, c]$. Thus, we have $\hat{u}(\sigma) = \lim_{n \rightarrow \infty} u_n(\sigma) = u(\sigma)$. Hence $\hat{u}$ is a solution to $(\text{IVP}; \sigma, u(\sigma))$ on $[\sigma, d]$. Note that u is a solution to $(\text{IVP}; \tau, z)$ on $[\tau, \sigma]$. Since the function $v : [\tau, d] \rightarrow X$ defined by $v(t) = u(t)$ for $t \in [\tau, \sigma]$ and $v(t) = \hat{u}(t)$ for $t \in [\sigma, d]$ is a solution to $(\text{IVP}; \tau, z)$ on $[\tau, d]$, we have $T(\tau, z) > d$. Since $v(t) = u(t)$ for $t \in [\tau, d]$ by uniqueness, we observe that the sequence $\{u_n\}$ converges to u uniformly on $[\sigma, d]$ as $n \rightarrow \infty$. $\square$

PROPOSITION 8. *T is a continuous function from Ω into $\mathbf{R} \cup \{\infty\}$.*

PROOF. Let $(\tau, z) \in \Omega$ and let $\{(t_n, x_n)\}_{n \geq 1}$ be a sequence in Ω converging to (τ, z) . Let $\tau < d < T(\tau, z)$. Since $\lim_{n \rightarrow \infty} (t_n, x_n) = (\tau, z)$, we deduce from Proposition 6 that $d < T(t_n, x_n)$ for sufficiently large integers

n . Thus $d \leq \liminf_{n \rightarrow \infty} T(t_n, x_n)$. Since d is arbitrary, we obtain $T(\tau, z) \leq \liminf_{n \rightarrow \infty} T(t_n, x_n)$. Note that

$$\tau < T(\tau, z) \leq \liminf_{n \rightarrow \infty} T(t_n, x_n) \leq \limsup_{n \rightarrow \infty} T(t_n, x_n),$$

and let d satisfy $\tau < d < \limsup_{n \rightarrow \infty} T(t_n, x_n)$. Then there exists a subsequence $\{(t_{n_k}, x_{n_k})\}_{k \geq 1}$ of $\{(t_n, x_n)\}_{n \geq 1}$ such that $d < T(t_{n_k}, x_{n_k})$ for $k \geq 1$. Since $(t_{n_k}, x_{n_k}) \rightarrow (\tau, z)$ as $k \rightarrow \infty$, it follows from Proposition 7 that $d < T(\tau, z)$. Since d is arbitrary chosen, we conclude that $\limsup_{n \rightarrow \infty} T(t_n, x_n) \leq T(\tau, z)$. Hence, we obtain $\lim_{n \rightarrow \infty} T(t_n, x_n) = T(\tau, z)$. $\square$

A global existence theorem is given as follows.

THEOREM 4. *Let C be a connected component of Ω and set $d = \sup\{t \in [a, b]; C(t) \neq \emptyset\}$. Then for each $(\tau, z) \in C$, (IVP; τ, z) has a unique solution on $[\tau, d)$ and the interval $[\tau, d)$ is the maximal interval of existence of solution. In particular, if Ω is connected, then for $(\tau, z) \in \Omega$, (IVP; τ, z) has a unique solution on $[\tau, b)$.*

PROOF. We shall show that $T : \Omega \rightarrow \mathbf{R} \cup \{\infty\}$ takes the constant value d on C . To prove that $T(C)$ is a singleton set, let $c, \hat{c} \in T(C) = \{T(t, x); (t, x) \in C\}$. Without loss of generality, we assume that $c \leq \hat{c}$, and set

$$C_1 = \{(t, x) \in C; T(t, x) \leq c\} \quad \text{and} \quad C_2 = \{(t, x) \in C; T(t, x) > c\}.$$

If $C = C_1$, then $\hat{c} \leq c$, and so $T(C)$ is a singleton set $\{c\}$. To prove that $C = C_1$, we have only to prove that $C_2 = \emptyset$ because C_1 and C_2 are disjoint. To this end, assume to the contrary that C_2 is nonempty. Since T is continuous on C by Proposition 8, C_2 is an open subset of C . Let $\{(t_n, x_n)\}_{n \geq 1}$ be a sequence in C_2 converging to $(t, x) \in C$. By the definition of C_2 , we have $c < T(t_n, x_n)$ for $n \geq 1$. Proposition 7 asserts that $c < T(t, x)$. This implies that C_2 is a closed subset of C . It follows that $C = C_1 \cup C_2$, and C_1 and C_2 are disjoint, nonempty and open in C . This is impossible because C is connected, and so we conclude that $C_2 = \emptyset$.

Since $T(C)$ is a singleton set, we can write $T(C) = \{c\}$ for some $c \in \mathbf{R} \cup \{\infty\}$. Since $t < T(t, x) = c$ for $(t, x) \in C$, we obtain $d = \sup\{t; C(t) \neq \emptyset\} \leq c$. On the other hand, let $s < c$. Note that $c = T(t, x)$ for some $(t, x) \in C$. If $t < s$ then a noncontinuable solution u to (IVP; t, x) satisfies $(s, u(s)) \in C$, and so $C(s) \neq \emptyset$. This implies that $s \leq d$. If $s \leq t$ then $s \leq t \leq d$ because $C(t) \neq \emptyset$. Since s is arbitrarily chosen such that $s < c$, we have $c \leq d$. Consequently, we get $T(C) = \{d\}$. $\square$

Theorem 1 is a consequence of Proposition 1 and Theorems 3 and 4.

4. Proof of Theorem 2

Proof of the necessity part. Let $(\tau, z) \in \Omega$ and $u(t) = U(t, \tau)z$ for $t \in [\tau, b)$. Let C be a connected component of Ω such that $(\tau, z) \in C$. Since $\{(t, u(t)); t \in [\tau, b)\}$ is a connected set in Ω containing (τ, z) , we have $(t, u(t)) \in C$ for $t \in [\tau, b)$ by the maximality of C ; hence $C(t) \neq \emptyset$ for $t \in [\tau, b)$. This means that (Q5) holds true. Since $u(\tau + h) \in \Omega(\tau + h)$ for $h \in (0, b - \tau)$, we have

$$\begin{aligned} h^{-1}d(z + hA(\tau, z), \Omega(\tau + h)) &\leq h^{-1}\|z + hA(\tau, z) - u(\tau + h)\| \\ &= \|A(\tau, u(\tau)) - h^{-1}(u(\tau + h) - u(\tau))\| \\ &\rightarrow \|A(\tau, u(\tau)) - u'(\tau)\| = 0 \end{aligned}$$

as $h \downarrow 0$. Thus, (Q3) also holds true. It remains to show that (Q4) holds true. We set

$$V_0(t, x, y) = \sup_{\sigma \in [t, b)} \left\{ \exp\left(-\int_t^\sigma \omega(\theta) d\theta\right) \|U(\sigma, t)x - U(\sigma, t)y\| \right\}$$

for $t \in [a, b)$ and $x, y \in \Omega(t)$. From (E1) and (E3) we see that

$$\|x - y\| \leq V_0(t, x, y) \leq L\|x - y\| \quad \text{for } t \in [a, b) \text{ and } x, y \in \Omega(t). \quad (4.1)$$

For any $x, y \in X$, $t \in [a, b)$ and $x', y' \in \Omega(t)$, we have

$$\begin{aligned} V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|) \\ \leq L\|x' - y'\| - L(\|x - x'\| + \|y - y'\|) \leq L\|x - y\|. \end{aligned}$$

Thus, we can define $V : [a, b) \times X \times X \rightarrow [0, \infty)$ by

$$V(t, x, y) = \sup_{(x', y') \in \Omega(t) \times \Omega(t)} \{\max(0, V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|))\}$$

for $(t, x, y) \in [a, b) \times X \times X$. Since

$$\begin{aligned} V_0(t, x', y') &\leq V_0(t, x', x) + V_0(t, x, y) + V_0(t, y, y') \\ &\leq V_0(t, x, y) + L(\|x - x'\| + \|y - y'\|) \end{aligned}$$

for $t \in [a, b)$ and $(x, y), (x', y') \in \Omega(t) \times \Omega(t)$, we have $V(t, x, y) \leq V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. The converse inequality follows readily from the definition of V . Thus $V(t, x, y) = V_0(t, x, y)$ for $t \in [a, b)$ and $(x, y) \in \Omega(t) \times \Omega(t)$. This combined with (4.1) implies that the functional V satisfies $(V4)'$ and $(V2)$.

Let $(x, y), (\hat{x}, \hat{y}) \in X \times X$ and $t \in [a, b)$. For any $(x', y') \in \Omega(t) \times \Omega(t)$, we have

$$\begin{aligned}
& V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|) \\
& \quad - (V_0(t, x', y') - L(\|\hat{x} - x'\| + \|\hat{y} - y'\|)) \\
& = L(\|\hat{x} - x'\| + \|\hat{y} - y'\|) - L(\|x - x'\| + \|y - y'\|) \\
& \leq L(\|\hat{x} - x\| + \|\hat{y} - y\|),
\end{aligned}$$

which implies that

$$V_0(t, x', y') - L(\|x - x'\| + \|y - y'\|) \leq V(t, \hat{x}, \hat{y}) + L(\|\hat{x} - x\| + \|\hat{y} - y\|)$$

and

$$V(t, x, y) \leq V(t, \hat{x}, \hat{y}) + L(\|\hat{x} - x\| + \|\hat{y} - y\|).$$

Thus, we obtain (V1).

To prove (V3), let $t_n \in [a, b)$ with $t_n \rightarrow t \in [a, b)$ as $n \rightarrow \infty$ and let $(x_n, y_n) \in \Omega(t_n) \times \Omega(t_n)$ with $(x_n, y_n) \rightarrow (x, y) \in \Omega(t) \times \Omega(t)$ as $n \rightarrow \infty$. Let $\sigma \in (t, b)$ and N a number such that $\sigma > t_n$ for $n \geq N$. Then we have

$$V_0(t_n, x_n, y_n) \geq \exp\left(-\int_{t_n}^{\sigma} \omega(\theta) d\theta\right) \|U(\sigma, t_n)x_n - U(\sigma, t_n)y_n\| \quad \text{for } n \geq N.$$

Taking the inferior limit as $n \rightarrow \infty$, we have

$$\liminf_{n \rightarrow \infty} V_0(t_n, x_n, y_n) \geq \exp\left(-\int_t^{\sigma} \omega(\theta) d\theta\right) \|U(\sigma, t)x - U(\sigma, t)y\|.$$

By (4.1), we have $V_0(t_n, x_n, y_n) \geq \|x_n - y_n\|$ for $n \geq 1$. Taking the inferior limit as $n \rightarrow \infty$, we see that the above inequality is also valid for $\sigma = t$. Thus, we have

$$\liminf_{n \rightarrow \infty} V_0(t_n, x_n, y_n) \geq V_0(t, x, y).$$

Finally, we prove the dissipativity condition

$$D_+ V(t, x, y)(A(t, x), A(t, y)) \leq \omega(t) V(t, x, y) \quad \text{for } x, y \in \Omega(t) \text{ and } t \in [a, b).$$

For this purpose, let $t \in [a, b)$ and $x, y \in \Omega(t)$. Since

$$\begin{aligned}
& \|U(\sigma, t+h)U(t+h, t)x - U(\sigma, t+h)U(t+h, t)y\| \\
& = \exp\left(\int_t^{\sigma} \omega(\theta) d\theta\right) \cdot \exp\left(-\int_t^{\sigma} \omega(\theta) d\theta\right) \|U(\sigma, t)x - U(\sigma, t)y\| \\
& \leq \exp\left(\int_t^{\sigma} \omega(\theta) d\theta\right) V_0(t, x, y) \\
& = \exp\left(\int_t^{t+h} \omega(\theta) d\theta\right) \cdot \exp\left(\int_{t+h}^{\sigma} \omega(\theta) d\theta\right) V_0(t, x, y)
\end{aligned}$$

for $h \in (0, b - t)$ and $\sigma \in [t + h, b)$, we have

$$V_0(t + h, U(t + h, t)x, U(t + h, t)y) \leq \exp\left(\int_t^{t+h} \omega(\theta) d\theta\right) V_0(t, x, y) \quad (4.2)$$

for $h \in (0, b - t)$. Since $V(t, x, y) = V_0(t, x, y)$ for $t \in [a, b)$ and $x, y \in \Omega(t)$ and since $V(t, \cdot, \cdot)$ is Lipschitz continuous on $X \times X$ with Lipschitz constant L , by (4.2) we have

$$\begin{aligned} & (V(t + h, x + hA(t, x), y + hA(t, y)) - V(t, x, y))/h \\ & \leq (V(t + h, U(t + h, t)x, U(t + h, t)y) - V(t, x, y))/h \\ & \quad + L(\|x + hA(t, x) - U(t + h, t)x\| + \|y + hA(t, y) - U(t + h, t)y\|)/h \\ & \leq \frac{1}{h} \left(\exp\left(\int_t^{t+h} \omega(\theta) d\theta\right) - 1 \right) V(t, x, y) \\ & \quad + L(\|x + hA(t, x) - U(t + h, t)x\| + \|y + hA(t, y) - U(t + h, t)y\|)/h \\ & \rightarrow \omega(t)V(t, x, y) \quad \text{as } h \downarrow 0. \end{aligned}$$

This means that the desired dissipativity condition holds true. $\square$

Proof of the sufficiency part. By condition (Q5), Theorem 4 asserts that for any $(\tau, z) \in \Omega$, there exists a unique global solution $u = u(\cdot; \tau, z)$ to (IVP; τ, z) on $[\tau, b)$. Define $\{U(t, \tau)\}_{(t, \tau) \in \mathcal{A}}$ by $U(t, \tau)z = u(t; \tau, z)$ for $(\tau, z) \in \Omega$ and $t \in [\tau, b)$. Then we see that for each $(t, \tau) \in \mathcal{A}$, $U(t, \tau)$ maps $\Omega(\tau)$ to $\Omega(t)$. We immediately obtain (E1) from the uniqueness of solutions to initial value problem (IVP; τ, z). By Proposition 1, we find, noting $(V4)'$, that

$$\begin{aligned} \|U(t, \tau)z - U(t, \tau)\hat{z}\| & \leq V(t, U(t, \tau)z, U(t, \tau)\hat{z}) \\ & \leq \exp\left(\int_\tau^t \omega(\theta) d\theta\right) V(\tau, z, \hat{z}) \leq L \exp\left(\int_\tau^t \omega(\theta) d\theta\right) \|z - \hat{z}\| \end{aligned}$$

for $z, \hat{z} \in \Omega(\tau)$ and $(t, \tau) \in \mathcal{A}$, namely, (E3) holds true.

It remains to show that (E2) holds true. Let $(t_n, \tau_n), (t, \tau) \in \mathcal{A}$, $z_n \in \Omega(\tau_n)$ and $z \in \Omega(\tau)$ and suppose that $(t_n, \tau_n) \rightarrow (t, \tau)$ and $z_n \rightarrow z$ as $n \rightarrow \infty$. We have to show that $u(t_n; \tau_n, z_n) = U(t_n, \tau_n)z_n \rightarrow u(t; \tau, z) = U(t, \tau)z$ as $n \rightarrow \infty$. First, we assume that $t > \tau$. Let $d \in (\tau, b)$ be a number such that $t < d$ and take $\sigma \in (\tau, t)$. Since $t_n \rightarrow t$ as $n \rightarrow \infty$, we may assume that $t_n \in [\sigma, d]$ for $n \geq 1$. Then, we deduce from Proposition 7 that $\lim_{n \rightarrow \infty} u(\cdot; \tau_n, z_n) = u(\cdot; \tau, z)$ uniformly on $[\sigma, d]$, and hence $u(t_n; \tau_n, z_n) \rightarrow u(t; \tau, z)$ as $n \rightarrow \infty$. Next, we assume that $t = \tau$. Since $u(t; \tau, z) = U(t, \tau)z = z$, we need to show that

$u(t_n; \tau_n, z_n) \rightarrow z$ as $n \rightarrow \infty$. To this end, let $M > 0$ and $R > 0$ be numbers such that $\tau + R < b$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_R(\tau, z)$. Since $(\tau_n, z_n) \rightarrow (\tau, z)$ as $n \rightarrow \infty$, there exists an integer $N \geq 1$ such that $\tau_n + R/2 < b$ and $(\tau_n, z_n) \in S_{R/2}(\tau, z)$ for $n \geq N$. Take $r = R/2$. Thus, we observe that if $n \geq N$, then $S_r(\tau_n, z_n) \subset S_R(\tau, z)$ and $\|A(s, y)\| \leq M$ for $(s, y) \in \Omega \cap S_r(\tau_n, z_n)$. Let $\sigma \in (0, r/(M+1))$. Thus, we deduce from Theorem 3 that if $n \geq N$ then

$$\|u(s; \tau_n, z_n) - u(\hat{s}; \tau_n, z_n)\| \leq M|s - \hat{s}|$$

for $s, \hat{s} \in [\tau_n, \tau_n + \sigma]$. Since $\tau_n \rightarrow \tau$ and $t_n \rightarrow t = \tau$ as $n \rightarrow \infty$, we find that $t_n \in [\tau_n, \tau_n + \sigma]$ for sufficient large n , and so the above inequality implies that

$$\|u(t_n; \tau_n, z_n) - z_n\| \leq M|t_n - \tau_n|$$

for sufficient large n . Since $z_n \rightarrow z$ as $n \rightarrow \infty$, we conclude that $u(t_n; \tau_n, z_n) \rightarrow z$ as $n \rightarrow \infty$. $\square$

5. Application to wave equations

In this section, we apply Theorem 1 to the initial value problem for non-linear wave equation with dissipation:

$$\begin{cases} \partial_t u = \partial_x v, & \partial_t v = \partial_x \sigma(t, u) - \gamma v, \\ u(0, x) = u_0(x), v(0, x) = v_0(x) & \text{for } x \in \mathbf{R} \text{ and } t \in [0, \infty). \end{cases} \quad (5.1)$$

Here γ is a positive constant and $\sigma(\cdot, \cdot)$ a real-valued smooth function on $[0, \infty) \times \mathbf{R}$ satisfying $\sigma(t, 0) = 0$ for $t \in [0, \infty)$. We make the following assumptions on the function σ .

- (i) There exists a positive constant δ_0 such that $\sigma_r(t, r) \geq \delta_0$ for $(t, r) \in [0, \infty) \times \mathbf{R}$.
- (ii) There exists a constant $L_0 > 0$ such that

$$\begin{aligned} \|\sigma_r(t, \cdot)\|_{L^\infty} &\leq L_0, & \|\sigma_{rr}(t, \cdot)\|_{L^\infty} &\leq L_0 \\ \text{and } \|\sigma_{rrr}(t, \cdot)\|_{L^\infty} &\leq L_0 & \text{for } t \in [0, \infty). \end{aligned}$$

- (iii) There exists a continuous integrable function $h : [0, \infty) \rightarrow [0, \infty)$ such that

$$\|\sigma_{tr}(t, \cdot)\|_{L^\infty} \leq h(t) \quad \text{for } t \in [0, \infty).$$

Let $X = L^2(\mathbf{R}) \times L^2(\mathbf{R})$ with the standard norm $\|(u, v)\| = (\|u\|_{L^2}^2 + \|v\|_{L^2}^2)^{1/2}$, and define $H : [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R}) \rightarrow \mathbf{R}$ by

$$\begin{aligned}
H(t, u, v) &= H^{(0)}(t, u, v) + H^{(1)}(t, u, v) + H^{(2)}(t, u, v) \\
&= \int_{-\infty}^{\infty} \left(\int_0^u \sigma(t, r) dr + \frac{1}{2} v^2 \right) dx \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} (\sigma_r(t, u) (\partial_x u)^2 + (\gamma u + \partial_x v)^2) dx \\
&\quad + \frac{1}{2} \int_{-\infty}^{\infty} (\sigma_r(t, u) (\partial_x^2 u)^2 + (\gamma \partial_x u + \partial_x^2 v)^2) dx
\end{aligned}$$

for $(u, v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ and $t \in [0, \infty)$. The assumptions imply that there exist constants $C_0 \geq c_0 > 0$ such that

$$c_0 \|(u, v)\|_{H^2 \times H^2}^2 \leq H(t, u, v) \leq C_0 \|(u, v)\|_{H^2 \times H^2}^2 \quad (5.2)$$

for $(u, v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ and $t \in [0, \infty)$. The following proposition will be used in order to convert the problem (5.1) into the initial value problem for a continuous mapping $A : \Omega \subset [0, \infty) \times X \rightarrow X$.

PROPOSITION 9. *Let $t \in [0, \infty)$ and $(u_0, v_0) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Then there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$, the problem*

$$(u_\lambda - u_0)/\lambda = \partial_x v_\lambda, \quad (5.3)$$

$$(v_\lambda - v_0)/\lambda = \sigma_r(t, u_0) \partial_x u_\lambda - \gamma v_\lambda \quad (5.4)$$

has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbf{R}) \times H^3(\mathbf{R})$ satisfying the following properties:

- (i) The family $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $\lambda \downarrow 0$.
- (ii) There exists a nondecreasing continuous function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$, depending only γ and $\sigma(\cdot, \cdot)$, such that

$$\begin{aligned}
&\frac{1}{\lambda} (H(t + \lambda, u_\lambda, v_\lambda) - H(t, u_0, v_0)) \\
&\leq \frac{1}{2\lambda} \left(\int_t^{t+\lambda} h(s) ds \right) \|u_\lambda\|_{H^2}^2 - \gamma \delta_0 \|\partial_x u_\lambda\|_{H^1}^2 \\
&\quad + (1 + \lambda^2) g(\|(u_0, v_0)\|_{H^2 \times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2}) \\
&\quad \times (\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2
\end{aligned} \quad (5.5)$$

for $\lambda \in (0, \lambda_0]$.

Here and subsequently, we use notation $a \vee b = \max\{a, b\}$ for $a, b \in \mathbf{R}$.

PROOF. Let $t \in [0, \infty)$ and $(u_0, v_0) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Define $D(L(t)) = H^1(\mathbf{R}) \times H^1(\mathbf{R})$ and

$$L(t)(u, v) = (\partial_x v, \sigma_r(t, u_0) \partial_x u - \gamma v)$$

for $(u, v) \in D(L(t))$. Let β_0 be a positive number such that $\beta_0 \geq L_0 \|\partial_x u_0\|_{L^\infty} / (2\sqrt{\delta_0})$. Since

$$\frac{\|\partial_x(\sigma_r(t, u_0))\|_{L^\infty}}{2\sqrt{\delta_0}} = \frac{\|\sigma_{rr}(t, u_0)\partial_x u_0\|_{L^\infty}}{2\sqrt{\delta_0}} \leq \beta_0,$$

we deduce from [8, Proposition 5.7] that $L(t) - \beta_0 I$ is m -dissipative in $X = L^2(\mathbf{R}) \times L^2(\mathbf{R})$ with inner product $((u, v), (\hat{u}, \hat{v})) = (\int_{-\infty}^{\infty} \sigma_r(t, u_0) u \hat{u} + v \hat{v} dx)^{1/2}$ for $(u, v), (\hat{u}, \hat{v}) \in X$. Choose $\lambda_0 > 0$ so that $\lambda_0 \beta_0 < 1$. Then, for $\lambda \in (0, \lambda_0]$, $(u_\lambda, v_\lambda) := (I - \lambda L(t))^{-1}(u_0, v_0)$ satisfies (5.3) and (5.4). Note that $D(L(t)^k) = H^k(\mathbf{R}) \times H^k(\mathbf{R})$ for $k = 2, 3$. It follows from the proof of [8, Proposition 5.7] that $(u_\lambda, v_\lambda) \in D(L(t)^3)$ and $L(t)^k(u_\lambda, v_\lambda) = (I - \lambda L(t))^{-1} L(t)^k(u_0, v_0)$ for $k = 0, 1, 2$ and that the family $\{L(t)^k(u_\lambda, v_\lambda)\}$ converges to $L(t)^k(u_0, v_0)$ in X as $\lambda \downarrow 0$, for $k = 0, 1, 2$. Hence the family $\{(u_\lambda, v_\lambda)\}$ converges to (u_0, v_0) in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $\lambda \downarrow 0$.

We shall show (ii). Since $\sigma(t, 0) = 0$, we have $\sigma(t, u_\lambda) \in H^1(\mathbf{R})$ and $\partial_x \sigma(t, u_\lambda) = \sigma_r(t, u_\lambda) \partial_x u_\lambda$. By (5.4), we get

$$\frac{1}{\lambda}(v_\lambda - v_0) = \partial_x \sigma(t, u_\lambda) - \gamma v_\lambda + (\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)) \partial_x u_\lambda.$$

We multiply this equality and (5.3) by v_λ and $\sigma(t, u_\lambda)$, respectively. The sum of these two equations gives us

$$\begin{aligned} & \frac{1}{\lambda} \sigma(t, u_\lambda)(u_\lambda - u_0) + \frac{1}{\lambda} v_\lambda(v_\lambda - v_0) \\ &= \partial_x(v_\lambda \sigma(t, u_\lambda)) - \gamma v_\lambda^2 + v_\lambda(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)) \partial_x u_\lambda. \end{aligned}$$

Integrating this equality, we have

$$\begin{aligned} & \frac{1}{\lambda} \int_{-\infty}^{\infty} \sigma(t, u_\lambda)(u_\lambda - u_0) dx + \frac{1}{\lambda} \int_{-\infty}^{\infty} v_\lambda(v_\lambda - v_0) dx \\ &= -\gamma \int_{-\infty}^{\infty} v_\lambda^2 dx + \int_{-\infty}^{\infty} v_\lambda(\sigma_r(t, u_0) - \sigma_r(t, u_\lambda)) \partial_x u_\lambda dx \\ &\leq \frac{1}{4\gamma} \int_{-\infty}^{\infty} (\sigma_r(t, u_0) - \sigma_r(t, u_\lambda))^2 (\partial_x u_\lambda)^2 dx \\ &\leq \frac{L_0^2}{4\gamma} \int_{-\infty}^{\infty} (u_0 - u_\lambda)^2 (\partial_x u_\lambda)^2 dx = \frac{\lambda^2 L_0^2}{4\gamma} \int_{-\infty}^{\infty} (\partial_x v_\lambda)^2 (\partial_x u_\lambda)^2 dx \\ &\leq \frac{\lambda^2 L_0^2}{4\gamma} \|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty} (\partial_x u_\lambda)^2 dx. \end{aligned}$$

Since the function $r \rightarrow \sigma(t, r)$ is nondecreasing, we have

$$\begin{aligned} & \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\int_{u_0}^{u_\lambda} \sigma(t, r) dr \right) dx + \frac{1}{2\lambda} \int_{-\infty}^{\infty} (v_\lambda^2 - v_0^2) dx \\ & \leq \frac{\lambda^2 L_0^2}{4\gamma} \|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty} (\partial_x u_\lambda)^2 dx, \end{aligned}$$

or

$$\begin{aligned} & \frac{1}{\lambda} (H^{(0)}(t + \lambda, u_\lambda, v_\lambda) - H^{(0)}(t, u_0, v_0)) \\ & \leq \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\int_0^{u_\lambda} (\sigma(t + \lambda, r) - \sigma(t, r)) dr \right) dx \\ & \quad + \frac{\lambda^2 L_0^2}{4\gamma} \|\partial_x v_\lambda\|_{H^1}^2 \int_{-\infty}^{\infty} (\partial_x u_\lambda)^2 dx. \end{aligned}$$

The first term on the right-hand side is estimated as follows:

$$\begin{aligned} & \frac{1}{\lambda} \int_{-\infty}^{\infty} \left(\int_0^{u_\lambda} (\sigma(t + \lambda, r) - \sigma(t, r)) dr \right) dx \\ & = \frac{1}{\lambda} \int_t^{t+\lambda} \left(\int_{-\infty}^{\infty} \left(\int_0^{u_\lambda} \sigma_t(s, r) dr \right) dx \right) ds \\ & = \frac{1}{\lambda} \int_t^{t+\lambda} \left(\int_{-\infty}^{\infty} \left(\int_0^{u_\lambda} \left(\int_0^1 \sigma_{tr}(s, \theta r) d\theta \right) r dr \right) dx \right) ds \\ & \leq \frac{1}{2\lambda} \left(\int_t^{t+\lambda} h(s) ds \right) \|u_\lambda\|_{L^2}^2. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{1}{\lambda} (H^{(0)}(t + \lambda, u_\lambda, v_\lambda) - H^{(0)}(t, u_0, v_0)) \\ & \leq \frac{1}{2\lambda} \left(\int_t^{t+\lambda} h(s) ds \right) \|u_\lambda\|_{L^2}^2 + \frac{\lambda^2}{4\gamma} L_0^2 \|\partial_x v_\lambda\|_{H^1}^2 \|\partial_x u_\lambda\|_{L^2}^2. \end{aligned} \quad (5.6)$$

Differentiating (5.3) and (5.4), we have

$$\frac{1}{\lambda} (\partial_x u_\lambda - \partial_x u_0) = \partial_x (\partial_x v_\lambda), \quad (5.7)$$

$$\frac{1}{\lambda} ((\gamma u_\lambda + \partial_x v_\lambda) - (\gamma u_0 + \partial_x v_0)) = \partial_x (\sigma_r(t, u_0) \partial_x u_\lambda). \quad (5.8)$$

We multiply (5.7) and (5.8) by $\sigma_r(t, u_0)\partial_x u_\lambda$ and $\gamma u_\lambda + \partial_x v_\lambda$, respectively. The sum of these two equations gives us

$$\begin{aligned} & \frac{1}{2\lambda} \sigma_r(t, u_0) ((\partial_x u_\lambda)^2 - (\partial_x u_0)^2) + \frac{1}{2\lambda} ((\gamma u_\lambda + \partial_x v_\lambda)^2 - (\gamma u_0 + \partial_x v_0)^2) \\ & \leq \partial_x (\sigma_r(t, u_0) \partial_x u_\lambda \partial_x v_\lambda) + \gamma u_\lambda \partial_x (\sigma_r(t, u_0) \partial_x u_\lambda). \end{aligned}$$

Integrating this equality, we have

$$\begin{aligned} & \frac{1}{2\lambda} \int_{-\infty}^{\infty} \sigma_r(t, u_0) ((\partial_x u_\lambda)^2 - (\partial_x u_0)^2) dx \\ & \quad + \frac{1}{2\lambda} \int_{-\infty}^{\infty} ((\gamma u_\lambda + \partial_x v_\lambda)^2 - (\gamma u_0 + \partial_x v_0)^2) dx \\ & \leq -\gamma \int_{-\infty}^{\infty} (\partial_x u_\lambda) (\sigma_r(t, u_0) \partial_x u_\lambda) dx. \end{aligned}$$

Thus

$$\begin{aligned} & \frac{1}{\lambda} (H^{(1)}(t + \lambda, u_\lambda, v_\lambda) - H^{(1)}(t, u_0, v_0)) \\ & \leq \frac{1}{2\lambda} \int_{-\infty}^{\infty} (\sigma_r(t + \lambda, u_\lambda) - \sigma_r(t, u_0)) (\partial_x u_\lambda)^2 dx - \gamma \int_{-\infty}^{\infty} \sigma_r(t, u_0) (\partial_x u_\lambda)^2 dx. \end{aligned}$$

Since

$$\begin{aligned} |\sigma_r(t + \lambda, u_\lambda) - \sigma_r(t, u_0)| & \leq |\sigma_r(t + \lambda, u_\lambda) - \sigma_r(t, u_\lambda)| + |\sigma_r(t, u_\lambda) - \sigma_r(t, u_0)| \\ & \leq \left| \int_t^{t+\lambda} \sigma_{tr}(s, u_\lambda) ds \right| + L_0 |u_\lambda - u_0| \leq \int_t^{t+\lambda} h(s) ds + \lambda L_0 |\partial_x v_\lambda|, \end{aligned} \quad (5.9)$$

we have

$$\begin{aligned} & \frac{1}{\lambda} (H^{(1)}(t + \lambda, u_\lambda, v_\lambda) - H^{(1)}(t, u_0, v_0)) \\ & \leq \frac{1}{2\lambda} \left(\int_t^{t+\lambda} h(s) ds \right) \|\partial_x u_\lambda\|_{L^2}^2 + \frac{1}{2} L_0 \|\partial_x v_\lambda\|_{H^1} \|\partial_x u_\lambda\|_{L^2}^2 \\ & \quad - \gamma \delta_0 \|\partial_x u_\lambda\|_{L^2}^2. \end{aligned} \quad (5.10)$$

Differentiating (5.7) and (5.8), we have

$$\frac{1}{\lambda} (\partial_x^2 u_\lambda - \partial_x^2 u_0) = \partial_x (\partial_x^2 v_\lambda), \quad (5.11)$$

$$\begin{aligned}
& \frac{1}{\lambda} ((\gamma \partial_x u_\lambda + \partial_x^2 v_\lambda) - (\gamma \partial_x u_0 + \partial_x^2 v_0)) \\
&= \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda + \sigma_r(t, u_0) \partial_x^2 u_\lambda). \tag{5.12}
\end{aligned}$$

We multiply (5.11) and (5.12) by $\sigma_r(t, u_0) \partial_x^2 u_\lambda$ and $\gamma \partial_x u_\lambda + \partial_x^2 v_\lambda$, respectively. The sum of these two equations gives us

$$\begin{aligned}
& \frac{1}{2\lambda} \sigma_r(t, u_0) ((\partial_x^2 u_\lambda)^2 - (\partial_x^2 u_0)^2) + \frac{1}{2\lambda} ((\gamma \partial_x u_\lambda + \partial_x^2 v_\lambda)^2 - (\gamma \partial_x u_0 + \partial_x^2 v_0)^2) \\
& \leq \partial_x (\sigma_r(t, u_0) \partial_x^2 u_\lambda \partial_x^2 v_\lambda) + \gamma \partial_x u_\lambda \partial_x (\sigma_r(t, u_0) \partial_x^2 u_\lambda) \\
& \quad + (\partial_x^2 v_\lambda + \gamma \partial_x u_\lambda) \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda).
\end{aligned}$$

Integrating this equality, we have

$$\begin{aligned}
& \frac{1}{2\lambda} \int_{-\infty}^{\infty} \sigma_r(t, u_0) ((\partial_x^2 u_\lambda)^2 - (\partial_x^2 u_0)^2) dx \\
& \quad + \frac{1}{2\lambda} \int_{-\infty}^{\infty} ((\gamma \partial_x u_\lambda + \partial_x^2 v_\lambda)^2 - (\gamma \partial_x u_0 + \partial_x^2 v_0)^2) dx \\
& \leq -\gamma \int_{-\infty}^{\infty} \sigma_r(t, u_0) (\partial_x^2 u_\lambda)^2 dx + \int_{-\infty}^{\infty} (\gamma \partial_x u_\lambda + \partial_x^2 v_\lambda) \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) dx \\
& = -\gamma \int_{-\infty}^{\infty} \sigma_r(t, u_0) (\partial_x^2 u_\lambda)^2 dx - \gamma \int_{-\infty}^{\infty} \partial_x^2 u_\lambda (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) dx \\
& \quad + \int_{-\infty}^{\infty} (\partial_x^2 v_\lambda) \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) dx.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{1}{\lambda} (H^{(2)}(t + \lambda, u_\lambda, v_\lambda) - H^{(2)}(t, u_0, v_0)) \\
& \leq \frac{1}{2\lambda} \int_{-\infty}^{\infty} (\sigma_r(t + \lambda, u_\lambda) - \sigma_r(t, u_0)) (\partial_x^2 u_\lambda)^2 dx - \gamma \int_{-\infty}^{\infty} \sigma_r(t, u_0) (\partial_x^2 u_\lambda)^2 dx \\
& \quad - \gamma \int_{-\infty}^{\infty} \partial_x^2 u_\lambda (\sigma_{rr}(t, u_0) (\partial_x u_0) \partial_x u_\lambda) dx \\
& \quad + \int_{-\infty}^{\infty} (\partial_x^2 v_\lambda) \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) dx. \tag{5.13}
\end{aligned}$$

The third term on the right-hand side is estimated by

$$\begin{aligned}
& -\gamma \int_{-\infty}^{\infty} \partial_x^2 u_\lambda (\sigma_{rr}(t, u_0) (\partial_x u_0) \partial_x u_\lambda) dx \\
& \leq \gamma L_0 \|\partial_x^2 u_\lambda\|_{L^2} \|\partial_x u_0\|_{L^\infty} \|\partial_x u_\lambda\|_{L^2} \leq \gamma L_0 \|u_0\|_{H^2} \|\partial_x u_\lambda\|_{H^1}^2.
\end{aligned}$$

Since

$$\begin{aligned}
& \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) \\
& = \sigma_{rrr}(t, u_0) (\partial_x u_0)^2 \partial_x u_\lambda + \sigma_{rr}(t, u_0) \partial_x^2 u_0 \partial_x u_\lambda + \sigma_{rr}(t, u_0) \partial_x u_0 \partial_x^2 u_\lambda,
\end{aligned}$$

we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} (\partial_x^2 v_\lambda) \partial_x (\sigma_{rr}(t, u_0) \partial_x u_0 \partial_x u_\lambda) dx \\
& \leq L_0 \|\partial_x^2 v_\lambda\|_{L^2} (\|\partial_x u_0\|_{L^\infty}^2 \|\partial_x u_\lambda\|_{L^2} \\
& \quad + \|\partial_x^2 u_0\|_{L^2} \|\partial_x u_\lambda\|_{L^\infty} + \|\partial_x u_0\|_{L^\infty} \|\partial_x^2 u_\lambda\|_{L^2}) \\
& \leq L_0 \|v_\lambda\|_{H^2} (\|u_0\|_{H^2} \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{L^2} \\
& \quad + \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{H^1} + \|\partial_x u_0\|_{H^1} \|\partial_x^2 u_\lambda\|_{L^2}) \\
& \leq L_0 \|v_\lambda\|_{H^2} (\|u_0\|_{H^2} + 2) \|\partial_x u_0\|_{H^1} \|\partial_x u_\lambda\|_{H^1}.
\end{aligned}$$

We estimate the first term on the right-hand side of (5.13) by (5.9), and combine the resulting inequality and the inequalities obtained above. This yields

$$\begin{aligned}
& \frac{1}{\lambda} (H^{(2)}(t + \lambda, u_\lambda, v_\lambda) - H^{(2)}(t, u_0, v_0)) \\
& \leq \frac{1}{2\lambda} \left(\int_t^{t+\lambda} h(s) ds \right) \|\partial_x^2 u_\lambda\|_{L^2}^2 + \frac{L_0}{2} \|\partial_x v_\lambda\|_{H^1} \|\partial_x^2 u_\lambda\|_{L^2}^2 - \gamma \delta_0 \|\partial_x^2 u_\lambda\|_{L^2}^2 \\
& \quad + L_0 (\gamma \|u_0\|_{H^2} + \|v_\lambda\|_{H^2} (\|u_0\|_{H^2} + 2)) (\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2.
\end{aligned}$$

Combining this inequality with (5.6) and (5.10) we observe that the desired inequality (5.5) is satisfied for the function

$$g(r) = L_0 r \left\{ \left(\frac{L_0 r}{4\gamma} \right) \vee (3 + \gamma + r) \right\} \quad \text{for } r \geq 0. \quad \square$$

Let c_0 be the constant in (5.2), and define $\hat{H} : [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R}) \rightarrow \mathbf{R}$ by

$$\hat{H}(t, u, v) = \exp\left(-\frac{1}{c_0} \int_0^t h(s) ds\right) H(t, u, v)$$

for $(t, u, v) \in [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Then we have

$$\hat{H}(t, u, v) \leq H(t, u, v) \leq \exp\left(\frac{1}{c_0} \int_0^\infty h(s) ds\right) \hat{H}(t, u, v) \quad (5.14)$$

for $(t, u, v) \in [0, \infty) \times H^2(\mathbf{R}) \times H^2(\mathbf{R})$. Since g is continuous and $g(0) = 0$, we choose a number $R_0 > 0$ so small that

$$\text{if } r \geq 0 \text{ and } r^2 \leq \frac{R_0}{c_0} \exp\left(\frac{1}{c_0} \int_0^\infty h(s) ds\right) \text{ then } g(r) < \gamma \delta_0, \quad (5.15)$$

and define a subset Ω of $[0, \infty) \times X$ by

$$\Omega = \{(t, (u, v)) \in [0, \infty) \times (H^2(\mathbf{R}) \times H^2(\mathbf{R})); \hat{H}(t, u, v) \leq R_0\}.$$

Let $r_0 = \sqrt{R_0/C_0}$, where C_0 is the constant in (5.2). Then, by (5.2) we have

$$S_0 := \{(u, v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); \|(u, v)\|_{H^2 \times H^2} \leq r_0\} \subset \Omega(t) \quad (5.16)$$

for any $t \in [0, \infty)$, and there exists a connected component C of Ω such that $[0, \infty) \times S_0 \subset C \subset \Omega$. Let R'_0 be the positive number such that $(R'_0)^2 = \frac{R_0}{c_0} \exp\left(\frac{1}{c_0} \int_0^\infty h(s) ds\right)$. Then, by (5.2) and (5.14) we have

$$\Omega(t) \subset S'_0 := \{(u, v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R}); \|(u, v)\|_{H^2 \times H^2} \leq R'_0\} \quad (5.17)$$

for any $t \in [0, \infty)$. Let V be the functional on $[0, \infty) \times X \times X$ defined by

$$V(t, (u, v), (\hat{u}, \hat{v})) = \left(\int_{-\infty}^\infty (\hat{v} - v)^2 + \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \right)^2 dx \right)^{1/2}$$

for $(u, v), (\hat{u}, \hat{v}) \in X$ and $t \in [0, \infty)$. It is easily seen that conditions (V1)–(V4) are satisfied. In particular, we see that for each $t \in [0, \infty)$, $V(t, \cdot, \cdot)$ is a metric on X and

$$\begin{aligned} \min\{1, \sqrt{\delta_0}\} \|(u, v) - (\hat{u}, \hat{v})\| &\leq V(t, (u, v), (\hat{u}, \hat{v})) \\ &\leq (1 \vee \sqrt{L_0}) \|(u, v) - (\hat{u}, \hat{v})\| \end{aligned}$$

for $(u, v), (\hat{u}, \hat{v}) \in X$. Consider the operator $A : \Omega \rightarrow X$ defined by

$$A(t, (u, v)) = (\partial_x v, \partial_x \sigma(t, u) - \gamma v)$$

for $(t, (u, v)) \in \Omega$. Then the nonlinear wave equation with dissipation (5.1) is converted into the initial value problem for A . We can prove that the initial value problem for A is globally well-posed, by Theorem 1 combined with the following theorem which will be proved by a sequence of propositions.

THEOREM 5. *The operator A satisfies $(\Omega 1)$ – $(\Omega 4)$.*

In view of (5.16) and (5.17), we are in a position to state the global solvability of the nonlinear wave equation with dissipation (5.1).

COROLLARY 1. *For any (u_0, v_0) such that $\|(u_0, v_0)\|_{H^2 \times H^2} \leq r_0$, there exists a unique time global solution $(u(\cdot), v(\cdot))$ to (5.1) such that*

$$(u(\cdot), v(\cdot)) \in C^1([0, \infty); L^2(\mathbf{R}) \times L^2(\mathbf{R})) \cap L^\infty(0, \infty; H^2(\mathbf{R}) \times H^2(\mathbf{R})).$$

REMARK 2. *Similar results are obtained in Yamada [23] and Matsumura [14].*

For the proof of Theorem 5 we follow the argument in [8]. We note here that

$$\|\partial_x w\|_{L^2}^2 \leq \|w\|_{L^2} \|\partial_x^2 w\|_{L^2} \quad \text{for } w \in H^2(\mathbf{R}). \quad (5.18)$$

PROPOSITION 10. *The operator A is continuous on Ω .*

PROOF. Let $(t, (u, v)), (\hat{t}, (\hat{u}, \hat{v})) \in \Omega$. Since $\sigma(t, 0) = 0$, we have

$$\sigma(t, u(x)) - \sigma(\hat{t}, u(x)) = u(x) \int_0^1 (\sigma_r(t, \hat{\theta}u(x)) - \sigma_r(\hat{t}, \hat{\theta}u(x))) d\hat{\theta}$$

and

$$\begin{aligned} & \|\sigma(t, u) - \sigma(\hat{t}, u)\|_{L^2}^2 \\ &= \int_{-\infty}^{\infty} \left((t - \hat{t})u(x) \int_0^1 \int_0^1 \sigma_{tr}(\hat{t} + \theta(t - \hat{t}), \hat{\theta}u(x)) d\theta d\hat{\theta} \right)^2 dx \\ &\leq \int_{-\infty}^{\infty} \left(|t - \hat{t}| \cdot |u(x)| \int_0^1 h(\hat{t} + \theta(t - \hat{t})) d\theta \right)^2 dx \\ &= \left(\int_{\hat{t}}^t h(s) ds \right)^2 \|u\|_{L^2}^2. \end{aligned}$$

Since $\|u\|_{L^2} \leq R'_0$ by (5.17) and $\|\sigma_r(\hat{t}, \cdot)\|_{L^\infty} \leq L_0$, we get

$$\begin{aligned} \|\sigma(t, u) - \sigma(\hat{t}, \hat{u})\|_{L^2} &\leq \|\sigma(t, u) - \sigma(\hat{t}, u)\|_{L^2} + \|\sigma(\hat{t}, u) - \sigma(\hat{t}, \hat{u})\|_{L^2} \\ &\leq \left| \int_{\hat{t}}^t h(s) ds \right| \|u\|_{L^2} + L_0 \|u - \hat{u}\|_{L^2} \\ &\leq R'_0 \left| \int_{\hat{t}}^t h(s) ds \right| + L_0 \|u - \hat{u}\|_{L^2}. \end{aligned}$$

By (5.17) we have $\|\partial_x^2(v - \hat{v})\|_{L^2} \leq \|\partial_x^2 v\|_{L^2} + \|\partial_x^2 \hat{v}\|_{L^2} \leq 2R'_0$. Since

$$\begin{aligned}\partial_x^2 \sigma(t, u(x)) &= \partial_x(\sigma_r(t, u(x))\partial_x u(x)) \\ &= \sigma_{rr}(t, u(x))(\partial_x u(x))^2 + \sigma_r(t, u(x))\partial_x^2 u(x),\end{aligned}$$

we get, by using the inequality $\|w\|_{L^\infty} \leq \|w\|_{H^1}$ for $w \in H^1(\mathbf{R})$,

$$\begin{aligned}\|\partial_x^2(\sigma(t, u) - \sigma(\hat{t}, \hat{u}))\|_{L^2} &\leq \|\partial_x^2 \sigma(t, u)\|_{L^2} + \|\partial_x^2 \sigma(\hat{t}, \hat{u})\|_{L^2} \\ &\leq L_0(\|(\partial_x u)^2\|_{L^2} + \|(\partial_x \hat{u})^2\|_{L^2}) + L_0(\|\partial_x^2 u\|_{L^2} + \|\partial_x^2 \hat{u}\|_{L^2}) \\ &\leq L_0(\|\partial_x u\|_{L^\infty} \|\partial_x u\|_{L^2} + \|\partial_x \hat{u}\|_{L^\infty} \|\partial_x \hat{u}\|_{L^2}) + 2L_0 R'_0 \\ &\leq 2L_0(R'_0)^2 + 2L_0 R'_0.\end{aligned}$$

Thus, using (5.18), we have

$$\begin{aligned}\|A(t, (u, v)) - A(\hat{t}, (\hat{u}, \hat{v}))\|^2 &\leq \|\partial_x(v - \hat{v})\|_{L^2}^2 + \|\partial_x(\sigma(t, u) - \sigma(\hat{t}, \hat{u})) - \gamma(v - \hat{v})\|_{L^2}^2 \\ &\leq \|\partial_x(v - \hat{v})\|_{L^2}^2 + 2\|\partial_x(\sigma(t, u) - \sigma(\hat{t}, \hat{u}))\|_{L^2}^2 + 2\gamma^2\|v - \hat{v}\|_{L^2}^2 \\ &\leq \|v - \hat{v}\|_{L^2} \|\partial_x^2(v - \hat{v})\|_{L^2} + 2\gamma^2\|v - \hat{v}\|_{L^2}^2 \\ &\quad + 2\|\sigma(t, u) - \sigma(\hat{t}, \hat{u})\|_{L^2} \|\partial_x^2(\sigma(t, u) - \sigma(\hat{t}, \hat{u}))\|_{L^2} \\ &\leq 2R'_0\|v - \hat{v}\|_{L^2} + 2\gamma^2\|v - \hat{v}\|_{L^2}^2 \\ &\quad + 4L_0 R'_0(1 + R'_0) \left(R'_0 \left| \int_{\hat{t}}^t h(s) ds \right| + L_0 \|u - \hat{u}\|_{L^2} \right),\end{aligned}$$

which implies the continuity of A on Ω . □

PROPOSITION 11. *Condition (Ω2) is satisfied for the set Ω .*

PROOF. Let $t_n \in [0, \infty)$ with $t_n \uparrow t \in [0, \infty)$ as $n \rightarrow \infty$. Let $(u, v) \in X$ and let $\{(u_n, v_n)\}$ be a sequence in X such that $(u_n, v_n) \in \Omega(t_n)$ for $n \geq 1$ and $(u_n, v_n) \rightarrow (u, v)$ in X as $n \rightarrow \infty$. We have to show that $(u, v) \in \Omega(t)$. Since the sequence $\{(u_n, v_n)\}$ is bounded in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ it follows that $(u, v) \in H^2(\mathbf{R}) \times H^2(\mathbf{R})$ and the sequence $\{(u_n, v_n)\}$ converges weakly to (u, v) in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $n \rightarrow \infty$. By (5.18), we see that the sequence $\{(u_n, v_n)\}$ converges to (u, v) in $H^1(\mathbf{R}) \times H^1(\mathbf{R})$ as $n \rightarrow \infty$. Moreover, $\{(u_n, v_n)\}$ converges to (u, v) in $L^\infty(\mathbf{R}) \times L^\infty(\mathbf{R})$ as $n \rightarrow \infty$. Since $\hat{H}(t_n, u_n, v_n) \leq R_0$ for $n \geq 1$, we have

$$\begin{aligned}
& R_0 \exp\left(\frac{1}{c_0} \int_0^{t_n} h(s) ds\right) \\
& \geq \int_{-\infty}^{\infty} \left(\int_0^{u_n} \sigma(t_n, r) dr + \frac{1}{2} v_n^2 \right) dx \\
& \quad + \frac{1}{2} \int_{-\infty}^{\infty} (\sigma_r(t_n, u_n) (\partial_x u_n)^2 + (\gamma u_n + \partial_x v_n)^2) dx \\
& \quad + \frac{1}{2} \int_{-\infty}^{\infty} (\sigma_r(t_n, u_n) (\partial_x^2 u_n)^2 + (\gamma \partial_x u_n + \partial_x^2 v_n)^2) dx \\
& = \int_{-\infty}^{\infty} \left(\int_0^{u_n} \sigma(t, r) dr + \frac{1}{2} v_n^2 \right) dx \\
& \quad + \frac{1}{2} \int_{-\infty}^{\infty} \{ \sigma_r(t, u) ((\partial_x u_n)^2 + (\partial_x^2 u_n)^2) + (\gamma u_n + \partial_x v_n)^2 + (\gamma \partial_x u_n + \partial_x^2 v_n)^2 \} dx \\
& \quad + \int_{-\infty}^{\infty} \left(\int_0^{u_n} (\sigma(t_n, r) - \sigma(t, r)) dr \right) dx \\
& \quad + \frac{1}{2} \int_{-\infty}^{\infty} \{ (\sigma_r(t_n, u_n) - \sigma_r(t, u)) ((\partial_x u_n)^2 + (\partial_x^2 u_n)^2) \} \quad \text{for } n \geq 1. \quad (5.19)
\end{aligned}$$

Since

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} \left(\int_0^{u_n} (\sigma(t_n, r) - \sigma(t, r)) dr \right) dx \right| \\
& = \left| \int_{-\infty}^{\infty} (t_n - t) \left(\int_0^{u_n} \left(\int_0^1 \int_0^1 \sigma_{tr}(t + \theta(t_n - t), \hat{\theta}r) d\theta d\hat{\theta} \right) r dr \right) dx \right| \\
& \leq \left| \int_{-\infty}^{\infty} (t_n - t) \left(\int_0^{u_n} \left(\int_0^1 h(t + \theta(t_n - t)) d\theta \right) r dr \right) dx \right| \\
& = \frac{\|u_n\|_{L^2}^2}{2} \left| \int_t^{t_n} h(s) ds \right|
\end{aligned}$$

and

$$\begin{aligned}
|\sigma_r(t_n, u_n) - \sigma_r(t, u)| & \leq |\sigma_r(t_n, u_n) - \sigma_r(t_n, u)| + |\sigma_r(t_n, u) - \sigma_r(t, u)| \\
& \leq L_0 \|u_n - u\|_{L^\infty} + \left| \int_t^{t_n} h(s) ds \right|
\end{aligned}$$

for $n \geq 1$, we have $R_0 \geq \hat{H}(t, u, v)$ by taking the inferior limit in (5.19) as $n \rightarrow \infty$. $\square$

PROPOSITION 12. *There exists a real-valued continuous function ω defined on $[0, \infty)$ such that*

$$D_+ V(t, (u, v), (\hat{u}, \hat{v}))(A(t, (u, v)), A(t, (\hat{u}, \hat{v}))) \leq \omega(t) V(t, (u, v), (\hat{u}, \hat{v}))$$

for $(u, v), (\hat{u}, \hat{v}) \in \Omega(t)$ and $t \in [0, \infty)$.

PROOF. Let $(u, v), (\hat{u}, \hat{v}) \in \Omega(t)$ for $t \in [0, \infty)$. Let $(\xi, \eta), (\hat{\xi}, \hat{\eta}) \in X$. Then we get

$$\begin{aligned} & 2D_+ V(t, (u, v), (\hat{u}, \hat{v}))((\xi, \eta), (\hat{\xi}, \hat{\eta})) V(t, (u, v), (\hat{u}, \hat{v})) \\ &= \liminf_{h \downarrow 0} \frac{1}{h} (V(t+h, (u, v) + h(\xi, \eta), (\hat{u}, \hat{v}) + h(\hat{\xi}, \hat{\eta}))^2 - V(t, (u, v), (\hat{u}, \hat{v}))^2) \\ &= \liminf_{h \downarrow 0} \frac{1}{h} \left\{ \int_{-\infty}^{\infty} ((\hat{v} + h\hat{\eta} - (v + h\eta))^2 - (\hat{v} - v)^2) dx \right. \\ &\quad \left. + \int_{-\infty}^{\infty} \left(\left(\int_{u+h\xi}^{\hat{u}+h\hat{\xi}} \sqrt{\sigma_r(t+h, r)} dr \right)^2 - \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \right)^2 \right) dx \right\} \\ &= \int_{-\infty}^{\infty} \left(2(\hat{v} - v)(\hat{\eta} - \eta) + 2 \int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \left\{ (\hat{\xi} \sqrt{\sigma_r(t, \hat{u})} - \xi \sqrt{\sigma_r(t, u)}) \right. \right. \\ &\quad \left. \left. + \int_u^{\hat{u}} \frac{\sigma_{tr}(t, r)}{2\sqrt{\sigma_r(t, r)}} dr \right\} \right) dx. \quad (5.20) \end{aligned}$$

Substituting $(\xi, \eta) = A(t, (u, v))$ and $(\hat{\xi}, \hat{\eta}) = A(t, (\hat{u}, \hat{v}))$ into (5.20) yields

$$\begin{aligned} & D_+ V(t, (u, v), (\hat{u}, \hat{v}))(A(t, (u, v)), A(t, (\hat{u}, \hat{v}))) V(t, (u, v), (\hat{u}, \hat{v})) \\ &= \int_{-\infty}^{\infty} \left((\hat{v} - v)(\partial_x(\sigma(t, \hat{u}) - \sigma(t, u)) - \gamma(\hat{v} - v)) \right. \\ &\quad \left. + \int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \left((\partial_x \hat{v} \sqrt{\sigma_r(t, \hat{u})} - \partial_x v \sqrt{\sigma_r(t, u)}) \right. \right. \\ &\quad \left. \left. + \int_u^{\hat{u}} \frac{\sigma_{tr}(t, r)}{2\sqrt{\sigma_r(t, r)}} dr \right) \right) dx \\ &= -\gamma \int_{-\infty}^{\infty} (\hat{v} - v)^2 dx - \int_{-\infty}^{\infty} \partial_x (\hat{v} - v)(\sigma(t, \hat{u}) - \sigma(t, u)) dx \end{aligned}$$

$$\begin{aligned}
& + \int_{-\infty}^{\infty} \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr ((\partial_x \hat{v} \sqrt{\sigma_r(t, \hat{u})} - \partial_x v \sqrt{\sigma_r(t, u)}) \right) dx \\
& + \int_{-\infty}^{\infty} \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \int_u^{\hat{u}} \frac{\sigma_{tr}(t, r)}{2\sqrt{\sigma_r(t, r)}} dr \right) dx \\
& = -\gamma \int_{-\infty}^{\infty} (\hat{v} - v)^2 dx + \int_{-\infty}^{\infty} \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \int_u^{\hat{u}} \frac{\sigma_{tr}(t, r)}{2\sqrt{\sigma_r(t, r)}} dr \right) dx \\
& + \int_{-\infty}^{\infty} \partial_x \hat{v} \int_u^{\hat{u}} (\sqrt{\sigma_r(t, r)} \sqrt{\sigma_r(t, \hat{u})} - \sigma_r(t, r)) dr dx \\
& + \int_{-\infty}^{\infty} \partial_x v \int_u^{\hat{u}} (\sqrt{\sigma_r(t, r)} \sqrt{\sigma_r(t, u)} - \sigma_r(t, r)) dr dx.
\end{aligned}$$

The second term on the right-hand side is estimated as follows:

$$\left| \int_{-\infty}^{\infty} \left(\int_u^{\hat{u}} \sqrt{\sigma_r(t, r)} dr \int_u^{\hat{u}} \frac{\sigma_{tr}(t, r)}{2\sqrt{\sigma_r(t, r)}} dr \right) dx \right| \leq \frac{\sqrt{L_0} h(t)}{2\sqrt{\delta_0}} \int_{-\infty}^{\infty} (\hat{u} - u)^2 dx.$$

The third and fourth terms are estimated as follows:

$$\begin{aligned}
& \left| \int_{-\infty}^{\infty} \partial_x \hat{v} \int_u^{\hat{u}} (\sqrt{\sigma_r(t, r)} \sqrt{\sigma_r(t, \hat{u})} - \sigma_r(t, r)) dr dx \right| \\
& \leq \|\partial_x \hat{v}\|_{L^\infty} \int_{-\infty}^{\infty} \left| \int_u^{\hat{u}} \frac{\sqrt{\sigma_r(t, r)} (\sigma_r(t, \hat{u}) - \sigma_r(t, r))}{\sqrt{\sigma_r(t, \hat{u})} + \sqrt{\sigma_r(t, r)}} dr \right| dx \\
& \leq L_0 \|\hat{v}\|_{H^2} \int_{-\infty}^{\infty} \left| \int_u^{\hat{u}} |\hat{u} - r| dr \right| dx = L_0 \|\hat{v}\|_{H^2} \|\hat{u} - u\|^2 / 2
\end{aligned}$$

and

$$\left| \int_{-\infty}^{\infty} \partial_x v \int_u^{\hat{u}} (\sqrt{\sigma_r(t, r)} \sqrt{\sigma_r(t, u)} - \sigma_r(t, r)) dr dx \right| \leq L_0 \|v\|_{H^2} \|\hat{u} - u\|^2 / 2.$$

Setting $\omega(t) = C'_0(1 + h(t))$ for a suitable positive number C'_0 , we conclude that

$$D_+ V(t, (u, v), (\hat{u}, \hat{v}))(A(t, (u, v)), A(t, (\hat{u}, \hat{v}))) \leq \omega(t) V(t, (u, v), (\hat{u}, \hat{v}))$$

for $(u, v), (\hat{u}, \hat{v}) \in \Omega(t)$ and $t \in [0, \infty)$. $\square$

PROPOSITION 13. For any $t \in [0, \infty)$ and $(u_0, v_0) \in \Omega(t)$,

$$\liminf_{\lambda \downarrow 0} \frac{1}{\lambda} d((u_0, v_0) + \lambda A(t, (u_0, v_0)), \Omega(t + \lambda)) = 0. \quad (5.21)$$

PROOF. Let $t \in [0, \infty)$ and $(u_0, v_0) \in \Omega(t)$. By (5.15) and (5.17), we note that

$$-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2}) < 0. \quad (5.22)$$

By Proposition 9, there exists $\lambda_0 > 0$ such that for any $\lambda \in (0, \lambda_0]$, the problem

$$\begin{cases} (u_\lambda - u_0)/\lambda = \partial_x v_\lambda, \\ (v_\lambda - v_0)/\lambda = \sigma_r(t, u_0) \partial_x u_\lambda - \gamma v_\lambda \end{cases}$$

has a solution $(u_\lambda, v_\lambda) \in H^3(\mathbf{R}) \times H^3(\mathbf{R})$ satisfying the properties (i) and (ii) in Proposition 9. If it is proved that $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficiently small $\lambda > 0$, then the subtangential condition (5.21) is shown to be satisfied by using the property (i) in Proposition 9.

We shall prove that $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficiently small $\lambda > 0$. By (5.2) and (5.5), we have

$$\begin{aligned} & \frac{1}{\lambda} \left(\left(1 - \frac{1}{2c_0} \int_t^{t+\lambda} h(s) ds \right) H(t + \lambda, u_\lambda, v_\lambda) - H(t, u_0, v_0) \right) \\ & \leq (1 + \lambda^2) g(\|(u_0, v_0)\|_{H^2 \times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2}) (\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2 \\ & \quad - \gamma\delta_0 \|\partial_x u_\lambda\|_{H^1}^2 \end{aligned} \quad (5.23)$$

for $\lambda \in (0, \lambda_0]$. Choose $\lambda_1 \in (0, \lambda_0]$ so that $\frac{1}{c_0} \int_t^{t+\lambda} h(s) ds \leq 1$ for $\lambda \in (0, \lambda_1]$ and $t \in [0, \infty)$. Noting that $e^{-2r} \leq 1 - r$ for $0 \leq r \leq 1/2$, we have

$$\exp\left(-\frac{1}{c_0} \int_t^{t+\lambda} h(s) ds\right) \leq 1 - \frac{1}{2c_0} \int_t^{t+\lambda} h(s) ds$$

for $\lambda \in (0, \lambda_1]$. Hence

$$\begin{aligned} & \frac{1}{\lambda} (\hat{H}(t + \lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) \\ & \leq \exp\left(-\frac{1}{c_0} \int_0^t h(s) ds\right) (-\gamma\delta_0 \|\partial_x u_\lambda\|_{H^1}^2 + (1 + \lambda^2) g(\|(u_0, v_0)\|_{H^2 \times H^2} \\ & \quad \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2}) (\|\partial_x u_0\|_{H^1} \vee \|\partial_x u_\lambda\|_{H^1})^2) \end{aligned} \quad (5.24)$$

for $\lambda \in (0, \lambda_1]$. Since $(u_\lambda, v_\lambda) \rightarrow (u_0, v_0)$ in $H^2(\mathbf{R}) \times H^2(\mathbf{R})$ as $\lambda \downarrow 0$, we have

$$\begin{aligned} & \limsup_{\lambda \downarrow 0} \frac{1}{\lambda} (\hat{H}(t + \lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) \\ & \leq \exp\left(-\frac{1}{c_0} \int_0^t h(s) ds\right) (-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2})) \|\partial_x u_0\|_{H^1}^2. \end{aligned} \quad (5.25)$$

If $\|\partial_x u_0\|_{H^1} \neq 0$, then we have $(-\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2}))\|\partial_x u_0\|_{H^2} < 0$ by (5.22). Hence (5.25) implies that $\hat{H}(t + \lambda, u_\lambda, v_\lambda) < \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficiently small $\lambda > 0$. If $\|\partial_x u_0\|_{H^1} = 0$, then (5.24) implies that

$$\begin{aligned} & \frac{1}{\lambda} (\hat{H}(t + \lambda, u_\lambda, v_\lambda) - \hat{H}(t, u_0, v_0)) \\ & \leq \exp\left(-\frac{1}{c_0} \int_0^t h(s) ds\right) \\ & \quad \times (-\gamma\delta_0 + (1 + \lambda^2)g(\|(u_0, v_0)\|_{H^2 \times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2}))\|\partial_x u_\lambda\|_{H^1}^2 \end{aligned}$$

for $\lambda \in (0, \lambda_1]$. Since

$$\begin{aligned} & \lim_{\lambda \downarrow 0} (-\gamma\delta_0 + (1 + \lambda^2)g(\|(u_0, v_0)\|_{H^2 \times H^2} \vee \|(u_\lambda, v_\lambda)\|_{H^2 \times H^2})) \\ & = -\gamma\delta_0 + g(\|(u_0, v_0)\|_{H^2 \times H^2}) < 0, \end{aligned}$$

the right-hand side of the above inequality is less than or equal to zero for sufficient small $\lambda > 0$; hence $\hat{H}(t + \lambda, u_\lambda, v_\lambda) \leq \hat{H}(t, u_0, v_0) \leq R_0$ and $(u_\lambda, v_\lambda) \in \Omega(t + \lambda)$ for sufficient small $\lambda > 0$. $\square$

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Anisohedral spherical triangles and classification of spherical tilings by congruent kites, darts and rhombi

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ABSTRACT. We classify all spherical monohedral (kite/dart/rhombus)-faced tilings, as follows: The set of spherical monohedral rhombus-faced tilings consists of (1) the central projection of the rhombic dodecahedron, (2) the central projection of the rhombic triacontahedron, (3) a series of non-isohedral tilings, and (4) a series of tilings which are topologically trapezohedra (here a trapezohedron is the dual of an antiprism.). The set of spherical tilings by congruent kites consists of (1) the central projection T of the tetragonal icosikaitetrahedron, (2) the central projection of the tetragonal hexacontahedron, (3) a non-isohedral tiling obtained from T by gliding a hemisphere of T with $\pi/4$ radian, and (4) a continuously deformable series of tilings which are topologically trapezohedra. The set of spherical tilings by congruent darts is a continuously deformable series of tilings which are topologically trapezohedra. In the above explanation, unless otherwise stated, the tilings we have enumerated are isohedral and admit no continuous deformation. We prove that if a spherical (kite/dart/rhombus) admits an edge-to-edge spherical monohedral tiling, then it also does a spherical isohedral tiling. We also prove that the set of anisohedral, spherical triangles (i.e., spherical triangles admitting spherical monohedral triangular tilings but not any spherical isohedral triangular tilings) consists of a certain, infinite series of isosceles triangles I , and an infinite series of right scalene triangles which are the bisections of I .

1. Introduction

Ueno-Agaoka [16] classified all spherical tilings by congruent triangles. Here, by a tiling, we mean an edge-to-edge tiling without two-valent vertices. This paper presents two applications of their classification.

As the first application, we prove that the set of *anisohedral*, spherical triangles consists of an infinite series of spherical isosceles triangles AI_n ($5 \neq n \geq 4$) and an infinite series of spherical right scalene triangles ARS_m ($m \geq 4$), each of which is the bisection of AI_m . See Figure 7 and Theorem 1. Here a tile T of a monohedral tiling is said to be *anisohedral*, provided that

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for every monohedral tiling $\mathcal{T}$ with the tile being congruent to T , $\mathcal{T}$ is not isohedral. A tiling on a constant curvature space is said to be *monohedral*, if the tiling consists of congruent tiles, while a tiling is said to be *isohedral* (or *tile-transitive*), if the symmetry group of the tiling acts transitively on the set of tiles. Monohedral tilings of the Euclidean space $\mathbf{R}^d$ have long been studied (see [10] for the survey). In his famous eighteenth problem, Hilbert posed three problems. The second problem of the three is

(†) “are there any anisohedral tile of the Euclidean space $\mathbf{R}^3$?” [12, 3].

As for the problem with the Euclidean space replaced by the two-dimensional sphere, the answer is yes. That is, there are anisohedral, spherical triangles, and they are the following: Recall that any spherical triangle is determined uniquely modulo congruence by the three inner angles [2, p. 62].

THEOREM 1. *The set of anisohedral, spherical triangles consists of*

- (1) *an infinite series of isosceles triangles AI_n such that the list of three inner angles is*

$$\left(\pi\left(\frac{1}{2} - \frac{1}{2n}\right), \pi\left(\frac{1}{2} - \frac{1}{2n}\right), \frac{2\pi}{n}\right) \quad (5 \neq n \geq 4); \quad \text{and}$$

- (2) *an infinite series of right scalene triangles ARS_m , which is the bisection of AI_m . The list of three inner angles of ARS_m is*

$$\left(\frac{\pi}{2}, \pi\left(\frac{1}{2} - \frac{1}{2m}\right), \frac{\pi}{m}\right) \quad (m \geq 4).$$

The second application of Ueno-Agaoka’s complete classification [16] of spherical monohedral triangular tilings is a complete classification (Theorem 2) of all spherical tilings by congruent kites, all spherical tilings by congruent darts, and all spherical tilings by congruent rhombi. Here, by a *kite* (*dart*, resp.), we mean a convex (non-convex, resp.) quadrangle such that the cyclic list of edge-lengths is $aabb$ ($a \neq b$). Figure 1 is the list of all spherical monohedral (kite/dart/rhombus)-faced tilings. Each tiling in a square frame is an instance of a continuously deformable tiling, while the other tilings admit no continuous deformation. The tilings left to the double strokes are sporadic, and the others are instances of series. All the tilings in Figure 1 are isohedral except a spherical monohedral kite-faced tiling gK_{24} (see the image in the topmost leftmost oval frame in Figure 1) and a spherical monohedral rhombus-faced tiling gTR_{4n+2} ($n \geq 2$; see the image in the bottom rightmost oval frame in Figure 1). However, the tile of the spherical monohedral, non-isohedral kite-faced tiling gK_{24} is congruent to the tile of a spherical isohedral kite-faced tiling K_{24} (see the image next to the image of gK_{24} in Figure 1), and the tile of

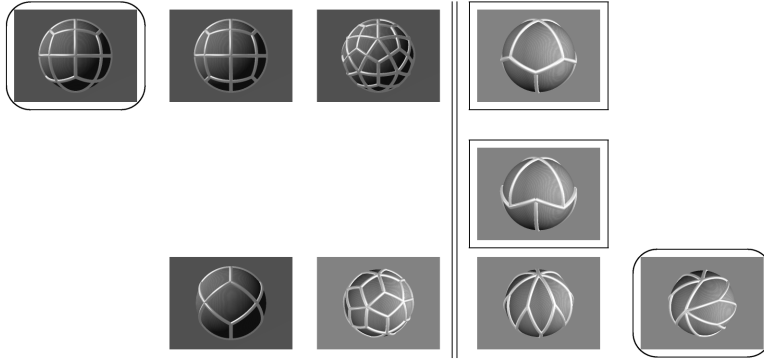


Fig. 1. The first line consists of spherical monohedral kite-faced tilings gK_{24} , K_{24} , K_{60} and $TR_8^{0.55}$. The second line is a spherical monohedral dart-faced tiling $TR_8^{0.45}$. The third line consists of spherical monohedral rhombus-faced tilings R_{12} , R_{30} , $TR_{14}^{6/7}$, and gTR_{14} . See Section 2 for the constructions of the tilings, and see Theorem 2 for the cyclic list of inner angles, the Schönflies symbol and the list of vertex types of each tiling.

the other spherical monohedral, non-isohedral rhombus-faced tiling gTR_{4n+2} is congruent to the tile of a spherical isohedral rhombus-faced tiling $TR_{4n+2}^{2n/(2n+1)}$ (see the image next to the image of gTR_{4n+2} in Figure 1). The definitions and the symbols for the tilings are given in the next section.

The consequences of the classification of spherical monohedral (kite/dart/rhombus)-faced tilings are: (1) none of spherical kites, spherical darts, and spherical rhombi is anisohedral. This is a slight refinement of the affirmative solution (Theorem 1) of Hilbert’s question (†) on an anisohedral tile in his eighteenth problem adapted for the two-dimensional sphere. (2) In order to classify all spherical monohedral quadrangular tilings, it is enough to classify those consisting of congruent quadrangles such that the cyclic list of edge-lengths of the tile is $aaab$ or $aabc$ (a , b , c are mutually unequal), because of Ueno-Agaoka [15]:

PROPOSITION 1. *In any spherical tilings consisting of congruent quadrangles, the tile necessarily has at least two equilateral adjacent edges.*

For an attempt to classify such spherical tilings, see [1].

The organization of this paper is as follows: In the next section, for each spherical monohedral (kite/dart/rhombus)-faced tilings appearing in Table 1, we give an explicit construction of it. Some constructions correspond to be “kis”ing [4, Chapter 21] for spherical tilings. This helps to show the isohedrality of spherical monohedral triangular tilings in Section 4. In Section 3, we state our main classification result (Theorem 2 including Table 1 and Figure 1). In Subsection 3.1 (Subsection 3.2, resp.), we prove that (1) Table 1 lists all

spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings, and (2) for each spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling $\mathcal{Q}$, we provide values to each inner angle of the graph, represent the symmetry of $\mathcal{Q}$ by a *Schönflies symbol* [5, p. 41], and decide whether the spherical (kite/dart)-faced (rhombus-faced, resp.) tiling $\mathcal{Q}$ is isohedral or not. In Subsection 3.2, we also determine the spherical coordinates of the vertices and the length of the edges of the spherical non-isohedral monohedral rhombus-faced tiling. In Section 4, we identify all the anisohedral, spherical triangles by using Proposition 2, and prove that there are no anisohedral, spherical (kites/darts/rhombi).

By the sphere, we mean the unit sphere $\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Unless otherwise stated, the unit for inner angles is π radian, while that for edge-length is radian. Ueno-Agaoka's classification (Table of [16]) of all spherical monohedral triangular tilings is included and is referred to as Table 2. The subscript of the name of a tiling indicates the number of the tiles, and the names of spherical monohedral triangular tilings are from Table 2. A spherical monohedral triangular tiling H_{20} in Table 2 is the central projection of the regular icosahedron to the sphere [16, p. 485]. Let $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) be the central projection of the regular octahedron (the regular dodecahedron, resp.) to the sphere.

2. Explicit constructions of the spherical monohedral (kite/dart/rhombus)-faced tilings

We will explicitly construct

- three spherical monohedral *kite-faced* tilings K_{60} , K_{24} , gK_{24} ,
- a deformable spherical monohedral (*kite/dart/rhombus*)-faced tiling TR_{2n}^α ($n \geq 3$) ($1/(2n) < \alpha < 1 - 1/(2n)$, $\alpha \neq 1/2$), and
- three spherical monohedral *rhombus-faced* tilings R_{12} , R_{30} , gTR_{4n+2} ($n \geq 2$).

We will construct the two spherical monohedral kite-faced tilings K_{60} and K_{24} , from two spherical tilings F_{120} and F_{48} by congruent right scalene triangles, and will construct the two spherical monohedral rhombus-faced tilings R_{12} and R_{30} , from two spherical tilings F_{24} and F_{60}^I by congruent isosceles triangles. Here F_{120} , F_{48} , F_{24} and F_{60}^I are listed in Ueno-Agaoka's complete classification ([16, Table], included as Table 2). According to [16, Figure 17 and p. 484], F_{24} is $\text{kis}(\mathcal{O}_8)$ and F_{60}^I is $\text{kis}(H_{20})$ where kis is the following construction.

DEFINITION 1. Let $\mathcal{T}$ be a spherical tiling. The “*kis*”ing of $\mathcal{T}$ is a spherical tiling by triangles obtained from $\mathcal{T}$ by joining the center of each

tile to each vertex of that tile. The resulting tiling is denoted by $\text{kis}(\mathcal{T})$. If $\mathcal{T}$ is a spherical tiling by n congruent rhombi, then $\text{kis}(\mathcal{T})$ is a spherical tiling by $4n$ congruent right scalene triangles. If $\mathcal{T}$ is a spherical tiling by n regular p -gons, then $\text{kis}(\mathcal{T})$ is a spherical tiling by pn congruent isosceles triangles.

The “kis”ing construction of spherical monohedral triangular tilings corresponds to the “kis”ing construction [4, Chapter 21] of *Archimedean duals*. In [4, Chapter 21], each Archimedean dual obtained via the “kis”ing of a Platonic solid or another Archimedean dual is named “kis...”

According to Ueno-Agaoka [16, Figure 21], a spherical tiling F_{120} (F_{48} , resp.) by congruent right scalene triangles, listed in Table 2, is constructed from $\mathcal{D}_{12}$ ($\mathcal{O}_8$, resp.), by (H) joining each center c of each face Φ of $\mathcal{D}_{12}$ ($\mathcal{O}_8$, resp.) to each vertex v of the face Φ , and (C) joining c to the midpoint m of each edge of Φ . Each edge (H) is the hypotenuse of the right scalene triangle of F_{120} (F_{48} , resp.). F_{120} (F_{48} , resp.) is a *barycentric subdivision* [8] (or also the *chamber system*) of $\mathcal{D}_{12}$ ($\mathcal{O}_8$, resp.).

2.1. K_{60} and K_{24} . A spherical monohedral kite-faced tiling K_{60} (K_{24} , resp.) is constructed from the spherical monohedral triangular tiling F_{120} (F_{48} , resp.), by deleting the hypotenuse (cf. (H)) of each right scalene triangular tile. In other words, K_{60} (K_{24} , resp.) is constructed from $\mathcal{D}_{12}$ ($\mathcal{O}_8$, resp.) by joining each center to the midpoints of the surrounding edges.

K_{24} is indeed a spherical monohedral *kite*-faced tiling, because of the following argument: All hypotenuses of the tiles of F_{48} are of equal length and bisect the inner angles $2/3$ between two adjacent edges obtained by (C), as well as an inner angle $1/2$ of $\mathcal{O}_8$. Hence, deleting the hypotenuses of the tiles of F_{48} results in a spherical monohedral tiling. The tile of K_{24} is a kite, because the tile has a pair $(2/3, 1/2)$ as opposite inner angles on the two vertices of the edge (H). However, the other pair of opposite inner angles is $(1/2, 1/2)$.

K_{60} is a spherical monohedral *kite*-faced tiling, by a similar argument.

We will prove that the symmetry of K_{24} is that of the regular octahedron, in order to assure that a construction (Subsection 2.2) of a spherical monohedral kite-faced tiling gK_{24} is well-defined. The symmetry of the regular octahedron (the regular icosahedron, resp.) is represented by a Schönflies symbol O_h (I_h , resp.) [5, p. 48 (p. 49, resp.)].

LEMMA 1. *The Schönflies symbol of K_{24} (K_{60} , resp.) is O_h (I_h , resp.).*

PROOF. Every symmetry operation σ of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) preserves the triangular (pentagonal, resp.) tiles, the centers of the triangular (pentagonal,

resp.) tiles and the midpoints of the edges of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.). Hence the Schönflies symbol of K_{24} (K_{60} , resp.) is O_h (I_h , resp.), by [5, p. 50]. $\square$

2.2. gK_{24} . A spherical monohedral tiling gK_{24} is constructed as follows: K_{24} has three mutually perpendicular, great circles. Each great circle consists of eight edges of K_{24} and is the intersection of K_{24} and some mirror plane of K_{24} . Choose one of the three great circles, and then a hemisphere $\mathcal{N}$ of K_{24} determined by the chosen great circle. Any choice of $\mathcal{N}$ results in a congruent spherical figure. It is because any great circle of K_{24} is transformed to any other great circle of K_{24} by some symmetry operation of K_{24} , by Lemma 1. Glide the hemisphere $\mathcal{N}$ by $\pi/4$ radian against the other hemisphere. Let the resulting spherical figure be gK_{24} . Because the boundary of the hemisphere $\mathcal{N}$ is equally subdivided into eight edges of K_{24} , the $\pi/4$ radian rotation keeps the tiling being edge-to-edge. See Figure 2. gK_{24} is a spherical monohedral *kite-faced* tiling, since the tiles are not changed by this gliding.

2.3. TR_{2n}^α . A spherical monohedral tiling TR_{2n}^α ($n \geq 3$) is constructed as in Figure 3 so that each pole is shared by n tiles. TR_{2n}^α is topologically the so-called *n-gonal trapezohedron of $2n$ faces* [11]. Here an *n-gonal trapezohedron* is the dual of an *n-gonal antiprism*. Let the cyclic list of four inner angles of the tile of TR_{2n}^α be $(\alpha, \beta, \alpha, \delta)$ such that $\beta = 2/n$ and $\delta = 2 - 2\alpha$, as in Figure 3 (left, right). Note $\alpha \neq 1/2$. Otherwise, $\delta = 1$ and thus the tile of TR_{2n}^α is a triangle.

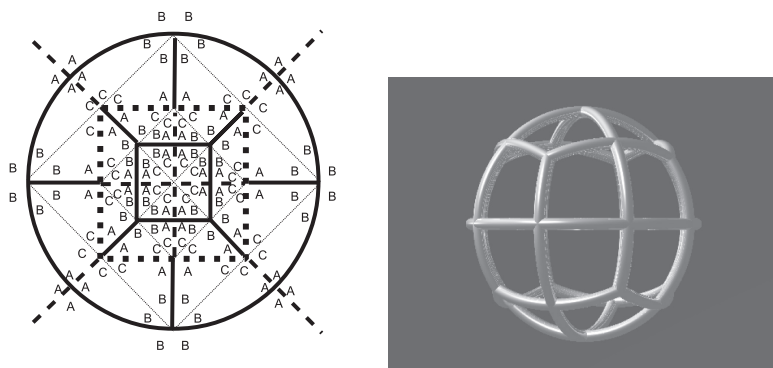


Fig. 2. The left is a graph of the spherical monohedral kite-faced tiling gK_{24} by 24 congruent kites (thick and dotted). The set of the thick, dotted, and thin edges represents a graph of the spherical monohedral triangular tiling TF_{48} . Cf. [16, Figure 88]. In gK_{24} , the inner angles between thick edges (thick edge and dotted edge, dotted edges, resp.) are $\beta = 2B$ ($\alpha = A$, $\delta = 2C$, resp.). The right is the view of gK_{24} from a two-fold axis of rotation. See Subsection 2.2.

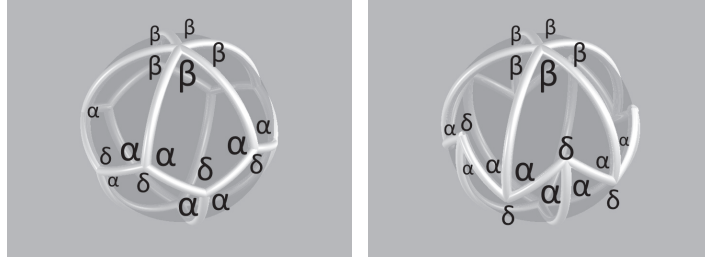


Fig. 3. The left figure is an explicit construction of a spherical monohedral kite-faced tiling TR_{2n}^α ($1/2 < \alpha < 1 - 1/(2n)$, $\alpha \neq 1 - 1/n$, $n \geq 3$), and the right figure is an explicit construction of a spherical monohedral dart-faced tiling TR_{2n}^α ($1/(2n) < \alpha < 1/2$, $n \geq 3$). The number of the tile is $2n$. The inner angle β is $2/n$.

LEMMA 2. *Let n be an integer greater than or equal to three. Then*

$$TR_{2n}^\alpha \text{ exists and the tile is a rhombus} \Leftrightarrow \alpha = 1 - \frac{1}{n}. \quad (1)$$

$$TR_{2n}^\alpha \text{ exists and the tile is a dart} \Leftrightarrow \frac{1}{2n} < \alpha < \frac{1}{2}. \quad (2)$$

$$TR_{2n}^\alpha \text{ exists and the tile is a kite} \Leftrightarrow \frac{1}{2} < \alpha < 1 - \frac{1}{2n} \text{ and } \alpha \neq 1 - \frac{1}{n}. \quad (3)$$

PROOF. First we prove that TR_{2n}^α exists as a spherical monohedral quadrangular tiling if and only if $1/(2n) < \alpha < 1 - 1/(2n)$ and $\alpha \neq 1/2$. Let $\mathcal{T}$ be obtained from TR_{2n}^α by drawing the meridian diagonal segment of each quadrangular tile Q of TR_{2n}^α . Then TR_{2n}^α exists as a spherical monohedral quadrangular tiling, if and only if $\mathcal{T}$ exists as a spherical monohedral triangular tiling. By Ueno-Agaoka's classification ([16, Table], included as Table 2) of all spherical monohedral triangular tilings and [16, Figure 4], the tiling $\mathcal{T}$ exists as a spherical monohedral triangular tiling, if and only if $\mathcal{T}$ is a spherical monohedral triangular tiling G_{4n} of Table 2 such that the list of inner angles of the triangular tile of G_{4n} is $(A, B, C) = (\alpha, \delta/2, \beta/2)$, if and only if $1/(2n) < \alpha < 1 - 1/(2n)$. Thus we have the desired consequence.

The equivalence (1) follows because $\beta = \delta$ if and only if the tile is a rhombus. The equivalence (3) follows because the tile is a kite if and only if all the inner angles α , β and δ are less than 1 and $\beta \neq \delta$. The equivalence (2) follows because $\delta = 2 - 2\alpha > 1$ if and only if the tile is a dart. $\square$

FACT 1. $TR_6^{2/3}$ is the central projection of the regular hexahedron to the sphere.

PROOF. In $TR_6^{2/3}$, the four inner angles of a tile Q are equal, and the horizontal diagonal segment of a tile and the vertical diagonal segment of the tile both bisect the inner angles of Q . Hence, the horizontal diagonal segment of Q subdivides Q into two congruent isosceles triangles T and the vertical diagonal segment of Q subdivides the tile into two isosceles triangles congruent to T . Thus the vertical diagonal segment and the horizontal diagonal segment are of the same length. Therefore Q is a regular square. $\square$

2.4. R_{12} and R_{30} . A spherical monohedral rhombus-faced tiling R_{12} (R_{30} , resp.) is constructed from the spherical monohedral triangular tiling F_{24} (F_{60}^I , resp.) mentioned in the beginning of this Section, by deleting the base edge of each isosceles triangular tile. Roughly speaking, R_{12} (R_{30} , resp.) is obtained from $\mathcal{O}_8$ (H_{20} , resp.) by replacing each edge with a rhombus.

2.5. gTR_{4n+2} ($n \geq 2$). We construct a spherical monohedral rhombus-faced tiling gTR_{4n+2} ($n \geq 2$), as follows: Consider $TR_{4n+2}^{2n/(2n+1)}$ (white edges, the top left image of Figure 4), and choose a hemisphere $\mathcal{W}$ (“western hemisphere”) such that on the boundary of the hemisphere $\mathcal{W}$, from the north pole N , we see a meridian edge, a meridian diagonal segment, another meridian edge and another meridian diagonal segment of $TR_{4n+2}^{2n/(2n+1)}$ (the second image of Figure 4). Every such choice results in a congruent hemisphere, because the polar axis of the tiling $TR_{4n+2}^{2n/(2n+1)}$ is a $(2n+1)$ -fold axis of rotation. Consider the copy of the hemisphere $\mathcal{W}$, rotate it around the polar axis by π radian (the third image of Figure 4). Let the resulting hemisphere be $\mathcal{W}'$.

Let η be the horizontal axis which passes through the center of the boundary of $\mathcal{W}'$ and is normal to the plane spanned by the boundary of $\mathcal{W}'$. Rotate $\mathcal{W}'$ around the axis η in the length of a meridian edge of $TR_{4n+2}^{2n/(2n+1)}$. Let $\mathcal{E}$ be the resulting hemisphere (“eastern hemisphere”). On the boundary of $\mathcal{E}$, we see meridian edges, and meridian diagonal segments, alternately. So the boundary of $\mathcal{E}$ corresponds to that of $\mathcal{W}$. Hence gluing $\mathcal{W}$ to $\mathcal{E}$ produces a spherical monohedral rhombus-faced tiling. Let the resulting tiling be gTR_{4n+2} (the white edges of the fourth image of Figure 4).

gTR_{4n+2} with $n = 1$ is $TR_6^{2/3}$ (cf. Fact 1).

REMARK 1. The K_{24} , K_{60} , R_{12} and R_{30} we have constructed are exactly the quadrangle-faced *Archimedean duals*, topologically. In the terminology of [4, Chapter 21], the corresponding quadrangle-faced Archimedean duals are the *tetragonal icosikaitetrahedron*, the *tetragonal hexacontahedron*, the *rhombic dodecahedron* and the *rhombic triacontahedron*. The last two Archimedean duals are studied deeply in [6, Sect. 4.7].

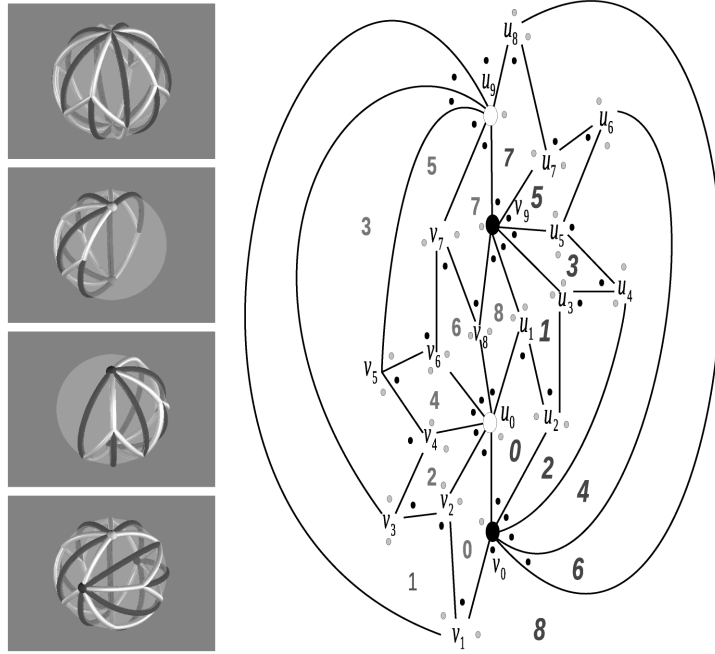


Fig. 4. The first column explains an explicit construction of a spherical monohedral rhombus-faced tiling gTR_{4n+2} (the white edges of the bottom image) from $TR_{4n+2}^{2n/(2n+1)}$ (the white edges of the top image), for $n = 2$. The western hemisphere $\mathcal{W}$ and the hemisphere $\mathcal{W}'$ mentioned in Subsection 2.5 are the second and the third image of the first column. The second column is a graph of gTR_{4n+2} ($n \geq 2$) for $n = 4$, where the inner angles designated by black circles are $\beta = 2/(2n + 1)$, and the other inner angles are $\alpha = 2n/(2n + 1)$. In the bottom image, we see two pairs of antipodal vertices of type $\alpha + (n + 1)\beta$. One is the pair of the north pole u_0 and the south pole u_{2n+1} , and the other is the pair of the vertex v_0 (the front black vertex) and the vertex v_{2n+1} (the back black vertex).

The spherical monohedral kite-faced tiling gK_{24} has the same skeleton as the dual of the *elongated square gyrobicupola* (or *pseudorhombicuboctahedron*). The pseudorhombicuboctahedron is the 37-th Johnson solids [13], [18] consisting of the eight regular triangles and the 18 regular squares. The vertex figures of this solid are the same, but the action of the symmetry group of the solid on the vertices is not transitive [14, p. 114]¹.

gTR_{4n+2} ($n \geq 2$) appears in [16, Figure 15]. gTR_{10} is the rightmost figure of the first line of [16, Figure 15].

¹ For every Archimedean solid, the faces are regular and the symmetry group acts transitively on the vertices. Prof. M. Deza let the second author know that a nickname of the pseudorhombicuboctahedron is the “14-th” Archimedean solid and he suggested that the nickname is due to Grünbaum [9].

3. Classification of spherical monohedral (kite/dart/rhombus)-faced tilings

We say a vertex of the tiling has *type* $n_1\alpha + n_2\beta + n_3\delta$, if around the vertex there are n_1 copies of the inner angle α , n_2 copies of the inner angle β , and n_3 copies of the inner angle δ .

THEOREM 2. (1) *The first column of Table 1 is the list of spherical monohedral kite-faced tilings, spherical monohedral dart-faced tilings, and spherical monohedral rhombus-faced tilings.*

(2) *The non-isohedral tilings in Table 1 are gK_{24} and gTR_{4n+2} ($n \geq 2$).*

For each spherical tiling listed in the first column of Table 1, the cyclic list of inner angles of the tile, the symmetry (Schönflies symbol) and the vertex types of the tiling are described in the second, the third, and the fourth columns, respectively.

(3) *The tilings appearing in Table 1 are non-congruent to each other.*

Name of spherical tiling	The cyclic list $(\alpha, \beta, \alpha, \delta)$ of inner angles of the tile [π radian]	Symmetry (Schönflies symbol)	Vertex types [the number]
K_{24}	$\alpha = \delta = \frac{1}{2}, \beta = \frac{2}{3}$	O_h	4α [12], 4δ [6], 3β [8]
gK_{24}	$\alpha = \delta = \frac{1}{2}, \beta = \frac{2}{3}$	D_{4d}	$\begin{cases} 4\alpha$ [8], 3β [8], $2\alpha + 2\delta$ [8], 4δ [2] \end{cases}
K_{60}	$\alpha = \frac{1}{2}, \beta = \frac{2}{3}, \delta = \frac{2}{5}$	I_h	$\begin{cases} 4\alpha$ [30], 3β [20], 5δ [12] \end{cases}
TR_{2n}^z ($\frac{1}{2} < \alpha < 1 - \frac{1}{2n}$, $\alpha \neq 1 - \frac{1}{n}$) ($n \geq 3$)	$\beta = \frac{2}{n}, 2\alpha + \delta = 2$	D_{nd}	$\begin{cases} 2\alpha + \delta$ [$2n$], $n\beta$ [2] \end{cases}
TR_{2n}^z ($\frac{1}{2n} < \alpha < \frac{1}{2}$) ($n \geq 3$)	$\beta = \frac{2}{n}, 2\alpha + \delta = 2$	D_{nd}	$\begin{cases} 2\alpha + \delta$ [$2n$], $n\beta$ [2] \end{cases}
R_{12}	$\alpha = \frac{1}{2}, \beta = \frac{2}{3}$	O_h	4α [6], 3β [8]
R_{30}	$\alpha = \frac{2}{5}, \beta = \frac{2}{3}$	I_h	3β [20], 5α [12]
$TR_{2n}^{1-1/n}$ ($n \geq 3$)	$\begin{cases} \alpha = \frac{n-1}{n}, \\ \beta = \frac{2}{n} \end{cases}$	$\begin{cases} O_h & (n=3) \\ D_{nd} & (n>3) \end{cases}$	$\begin{cases} 2\alpha + \beta$ [$2n$], $n\beta$ [2] \end{cases}
gTR_{4n+2} ($n \geq 2$)	$\begin{cases} \alpha = \frac{2n}{2n+1}, \\ \beta = \frac{2}{2n+1} \end{cases}$	D_2	$\begin{cases} \alpha + (n+1)\beta$ [4] $2\alpha + \beta$ [$4n$] \end{cases}

Table 1. A complete table of the spherical monohedral tilings consisting of (kites/darts/rhombi) (from top to bottom). The three parts are separated by double horizontal lines. The subscript of each tiling indicates the number of tiles. The spherical monohedral kite-faced tiling TR_{2n}^z and the spherical monohedral dart-faced tiling TR_{2n}^z are continuously deformable, and the tilings gK_{24} and gTR_{4n+2} are non-isohedral. For the Schönflies symbols D_2 (D_{nd} , O_h , I_h , resp.), see pp. 41–42 (p. 43, p. 48, p. 49, resp.) in [5]. The tilings appearing in the table are non-congruent to each other. See Theorem 2.

The rest of this section is devoted to the proof of Theorem 2.

As for Theorem 2 (1), the first column of Table 1 are exactly spherical monohedral (kite/dart/rhombus)-faced tilings which are constructed in Section 2. To show that Table 1 is *the* list of spherical monohedral (kite/dart/rhombus)-faced tilings, we combine Lemma 3, with Ueno-Agaoka's complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings.

DEFINITION 2. We say a spherical monohedral triangular tiling $\mathcal{T}$ *yields* a spherical monohedral quadrangular tiling $\mathcal{Q}$, if $\mathcal{T}$ has twice as many tiles as $\mathcal{Q}$, and the deletion of some edges of $\mathcal{T}$ results in $\mathcal{Q}$.

Then we have the following:

- LEMMA 3.** (1) *For every spherical tiling $\mathcal{Q}$ by congruent kites or by congruent darts, there is a spherical tiling $\mathcal{T}$ by F congruent non-equilateral isosceles triangles or by F congruent scalene triangles, such that $\mathcal{T}$ yields $\mathcal{Q}$ and $F \geq 12$.*
- (2) *For every spherical monohedral rhombus-faced tiling $\mathcal{R}$, there is some spherical tiling $\mathcal{T}$ by $F \geq 12$ congruent isosceles triangles T , such that $\mathcal{T}$ yields $\mathcal{R}$ and the two base inner angles of the triangle T is smaller than the other inner angle of T .*

PROOF. The number F of tiles of $\mathcal{T}$ is an even number greater than or equal to 12, because $\mathcal{Q}$ has more than six tiles by Euler's theorem (see the first part of the proof of [15, Proposition 1]).

(1) Each tile Q of the tiling $\mathcal{Q}$ has a vertex u incident to two edges of length a and a vertex v incident to two edges of length b such that $a \neq b$, because the tile Q is a kite or a dart. The diagonal segment between u and v subdivides Q into two congruent triangles T . T is a non-equilateral isosceles triangle or a scalene triangle, because T has an edge of length a and an edge of length b such that $a \neq b$. Hence, subdividing each tile Q of the tiling $\mathcal{Q}$ results in a spherical tiling by congruent non-equilateral isosceles triangles T or by congruent scalene triangles T . (2) Assume otherwise. Then there is a spherical monohedral rhombus-faced tiling $\mathcal{R}$ with the following condition: whenever a spherical tiling $\mathcal{T}$ by congruent isosceles triangles T yields $\mathcal{R}$, the two base angles of the isosceles triangle T is greater than or equal to the other inner angle. Now let the cyclic list of four inner angles of the rhombic tile R be $(\alpha, \beta, \alpha, \beta)$. A subdivision of R results in an isosceles triangle such that the list of inner angles is $(\alpha, \beta/2, \beta/2)$ or $(\beta, \alpha/2, \alpha/2)$. Moreover, $\beta/2 \geq \alpha$ and $\alpha/2 \geq \beta$. This implies $\alpha = \beta = 0$, which is a contradiction. $\square$

Name	V	E	inner angles A, B, C [π rad]	vertex type [number]
$\square F_4$	4	6	$A + B + C = 2,$ $1/2 < A, B, C < 1$	$A + B + C$ [4]
F_{12}^I	8	18	$A = 2/3, B = C = 1/3$	$3A$ [4], $6B$ [4]
F_{12}^{II}	8	18	$A = 2/3, B = C = 1/3$	$\begin{cases} 3A & [2], & 2A + 2B & [2] \\ A + 4B & [2], & 6B & [2] \end{cases}$
F_{12}^{III}	8	18	$A = 2/3, B = C = 1/3$	$\begin{cases} 3A & [1], & 2A + 2B & [3] \\ A + 4B & [3], & 6B & [1] \end{cases}$
F_{24}	14	36	$A = 2/3, B = C = 1/4$	$3A$ [8], $8B$ [6]
F_{48}	26	72	$A = 1/2, B = 1/3, C = 1/4$	$\begin{cases} 4A & [12], & 6B & [8] \\ 8C & [6] \end{cases}$
TF_{48}	26	72	$A = 1/2, B = 1/3, C = 1/4$	$\begin{cases} 4A & [8], & 6B & [8] \\ 8C & [2], & 2A + 4C & [8] \end{cases}$
F_{60}^I	32	90	$A = 2/3, B = C = 1/5$	$3A$ [20], $10B$ [12]
F_{60}^{II}	32	90	$A = 2/5, B = C = 1/3$	$5A$ [12], $6B$ [20]
F_{120}	62	180	$A = 1/2, B = 1/3, C = 1/5$	$\begin{cases} 4A & [30], & 6B & [20] \\ 10C & [12] \end{cases}$
$\square G_{4n} \ (n \geq 2)$	$2n + 2$	$6n$	$A + B = 1, C = 1/n,$ $\frac{1}{2n} < A, B < \frac{2n-1}{2n}$	$\begin{cases} 2A + 2B & [2n] \\ 2nC & [2] \end{cases}$
$G_{4n+2} \ (n \geq 1)$	$2n + 3$	$6n + 3$	$A = B = 1/2, C = \frac{2}{2n+1}$	$\begin{cases} 4A & [2n+1] \\ (2n+1)C & [2] \end{cases}$
$TG_{8n} \ (n \geq 2)$	$4n + 2$	$12n$	$A = B = 1/2, C = \frac{1}{2n}$	$\begin{cases} 4A & [4n-2] \\ 2A + 2nC & [4] \end{cases}$
$\square TG_{8n+4} \ (n \geq 1)$	$4n + 4$	$12n + 6$	$A + B = 1, C = \frac{1}{2n+1},$ $\frac{1}{4n+2} < A, B < \frac{4n+1}{4n+2}$	$\begin{cases} A + B + (2n+1)C & [4] \\ 2A + 2B & [4n] \end{cases}$
$MTG_{8n+4}^I \ (n \geq 1)$	$4n + 4$	$12n + 6$	$A = \frac{n+1}{2n+1}, B = \frac{n}{2n+1}, C = \frac{1}{2n+1}$	$\begin{cases} A + B + (2n+1)C & [2] \\ A + 3B + C & [2] \\ 2A + 2B & [4n-2] \\ 2A + 2nC & [2] \end{cases}$
$MTG_{8n+4}^{II} \ (n \geq 2)$	$4n + 4$	$12n + 6$	$A = \frac{n+1}{2n+1}, B = \frac{n}{2n+1}, C = \frac{1}{2n+1}$	$\begin{cases} A + 3B + C & [4] \\ 2A + 2B & [4n-4] \\ 2A + 2nC & [4] \end{cases}$
$H_{4n} \ (n \geq 3)$	$2n + 2$	$6n$	$A = B = \frac{n-1}{2n}, C = 2/n$	$4A + C$ [$2n$], nC [2]
$TH_{8n+4} \ (n \geq 3)$	$4n + 4$	$12n + 6$	$A = B = \frac{n}{2n+1}, C = \frac{2}{2n+1}$	$\begin{cases} 4A + C & [4n] \\ 2A + (n+1)C & [4] \end{cases}$
$I_{8n} \ (n \geq 3)$	$4n + 2$	$12n$	$A = 1/2, B = \frac{n-1}{2n}, C = 1/n$	$\begin{cases} 4A & [2n] \\ 4B + 2C & [2n] \\ 2nC & [2] \end{cases}$
$TI_{16n+8} \ (n \geq 2)$	$8n + 6$	$24n + 12$	$A = 1/2, B = \frac{n}{2n+1}, C = \frac{1}{2n+1}$	$\begin{cases} 4A & [4n+2] \\ 4B + 2C & [4n] \\ 2B + (2n+2)C & [4] \end{cases}$

Table 2. The classification of the spherical monohedral triangular tilings [16, Table]. The first column lists the names. The explicit constructions of the tilings are explained in [16, Section 2]. There are ten sporadic tilings (above the horizontal line) and ten series (below the horizontal line). The mark $\square$ indicates that the tiling is continuously deformable. None of these tilings are isomorphic to each other except the trivial case given by the exchange of $A + B + C = 2$ (F_4), $A + B = 1$ (G_{4n}, TG_{8n+4}).

We will prove all spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings appear in Table 1, by exhausting the pairs $(\mathcal{T}, \mathcal{Q})$ of Lemma 3 (1) (Lemma 3 (2), resp.) in Lemma 5 (Lemma 7, resp.), with Lemma 4 (Lemma 4 (I), resp.).

LEMMA 4. *Let $F \geq 12$ be an even number. Let $\mathcal{T}$ be a spherical tiling by F congruent isosceles triangles T . In Ueno-Agaoka's complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings, the following two assertions hold: Inside “ $\{\dots\}$ ”, the greatest and the smallest inner angles of the triangular tile T are given.*

(I) *If the two base angles of the spherical isosceles triangle T are smaller than the other inner angle of T , then the spherical monohedral triangular tiling $\mathcal{T}$ is one of the following:*

- $F_{12}^I, F_{12}^{II}, F_{12}^{III}, F_{24}, F_{60}^I, F_{60}^{II},$
- $G_{4n} \ (n \geq 3) \ \{1 - 1/n, 1/n\},$
- $TG_{8n+4} \ (n \geq 1) \ \{2n/(2n+1), 1/(2n+1)\},$
- $MTG_{12}^I \ \{2/3, 1/3\},$
- $I_{24} \ \{1/2, 1/3\},$
- $H_{12} \ \{2/3, 1/3\}, H_{16} \ \{1/2, 3/8\}.$

(II) *If the two base angles of the spherical isosceles triangle T are greater than the other inner angle, then the spherical monohedral triangular tiling $\mathcal{T}$ is one of the following:*

- $G_{4n+2} \ (n \geq 3) \ \{1/2, 2/(2n+1)\},$
- $TG_{8n} \ (n \geq 2) \ \{1/2, 1/(2n)\},$
- $H_{4n} \ (n \geq 6), TH_{8n+4} \ (n \geq 3),$
- $G_{4n} \ (n \geq 3) \ \{1/2, 1/n\},$
- $TG_{8n+4} \ (n \geq 1) \ \{1/2, 1/(2n+1)\}.$

PROOF. The lemma is [16, Proposition 6] where the number F is greater than or equal to 12. $\square$

In Subsubsection 3.1.2, to prove Theorem 2 (2), we will use the following:

FACT 2. *Let $\mathcal{T}$ be a spherical tiling by congruent triangles such that the list of three inner angles is (A, B, C) , and let $\mathcal{Q}$ be a spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling such that the cyclic list of four inner angles of the tile is*

$$(\alpha, \beta, \alpha, \delta) \quad ((\alpha, \beta, \alpha, \beta), \text{ resp.}).$$

If $\mathcal{T}$ yields $\mathcal{Q}$ and $(\alpha, \beta, \alpha, \delta) = (A, 2B, A, 2C) \ ((\alpha, \beta, \alpha, \beta) = (2B, A, 2B, A), \text{ resp.})$, then the vertex types of $\mathcal{Q}$ are exactly obtained from those of $\mathcal{T}$ by applying a substitution $(A := \alpha, B = \beta/2, C = \delta/2) \ ((A := \beta, B := \alpha/2), \text{ resp.})$.

Theorem 2 (3) follows from Theorem 2 (2), because spherical monohedral tilings in Table 1 are not congruent to each other by the form of the tiles (i.e., kites, darts, rhombi), by the symmetry, or by the number of the tiles.

3.1. A complete classification of spherical monohedral (kite/dart)-faced tilings.

3.1.1. The completeness (Theorem 2 (1)).

LEMMA 5. *Let $\mathcal{T}$ be a spherical tiling by congruent non-equilateral triangles T , and $\mathcal{Q}$ be a spherical kite-faced or dart-faced monohedral tiling. Then if $\mathcal{T}$ yields $\mathcal{Q}$, then $(\mathcal{T}, \mathcal{Q})$ is (F_{48}, K_{24}) , (TF_{48}, gK_{24}) , (F_{120}, K_{60}) or (G_{4n}, TR_{2n}^α) for some integer $n \geq 3$ and some $\alpha \in (1/(2n), 1 - 1/(2n)) \setminus \{1/2, 1 - 1/n\}$.*

To prove the previous lemma, we will use the following “uniqueness” lemma:

LEMMA 6. *Assume that $\mathcal{T}$ is a spherical tiling by congruent triangles T , $\mathcal{Q}$ is a spherical tiling by congruent quadrangles Q , and $\mathcal{T}$ yields $\mathcal{Q}$ in a manner that Q is a kite or a dart. Then the following two assertions hold:*

- (1) *$\mathcal{T}$ is none of entries TG_{8n+4} ($n \geq 1$), MTG_{8n+4}^I ($n \geq 2$), and MTG_{8n+4}^{II} ($n \geq 2$) of Table 2.*
- (2) *When T is a non-equilateral isosceles triangle, T is not rectangular. Moreover, the quadrangular tile Q is unique modulo congruence for the given spherical monohedral triangular tiling $\mathcal{T}$.*

PROOF. From the assumption, $\mathcal{T}$ yields $\mathcal{Q}$ in a manner that Q is a kite or a dart. So, the tile Q of $\mathcal{Q}$ has a pair $(2X, 2Y)$ of opposite inner angles for some inner angles X, Y of the triangular tile T . It is because we must glue two copies of the triangular tile T of $\mathcal{T}$ at the edges of the same length, to obtain the quadrangular tile Q of $\mathcal{Q}$. Thus there is an inner angle X of T such that the number of occurrences of X is even for any vertex type of $\mathcal{T}$.

According to Table 2, however, if $\mathcal{T}$ is TG_{8n+4} ($n \geq 1$) or MTG_{8n+4}^I ($n \geq 2$), then for every inner angle $X \in \{A, B, C\}$, the number of occurrences of X is odd in a vertex type $A + B + (2n + 1)C$ of $\mathcal{T}$. So TG_{8n+4} ($n \geq 1$) and MTG_{8n+4}^I ($n \geq 2$) are impossible. Similarly, for MTG_{8n+4}^{II} ($n \geq 2$), for every $X \in \{A, B, C\}$, the number of occurrences of X is odd in a vertex type $A + 3B + C$ of MTG_{8n+4}^{II} . So MTG_{8n+4}^{II} ($n \geq 2$) is impossible too. Thus the assertion (1) follows.

The first part of the assertion (2) is proved as follows. The list of three inner angles of the isosceles triangular tile T is (X, X, Y) without loss of

generality. Because the quadrangular tile Q is not a rhombus, the cyclic list of the four inner angles is $(X, 2Y, X, 2Y)$. If X or Y is $\pi/2$ radian, then some inner angle of Q is π radian, which is a contradiction. Hence, the triangular tile of $\mathcal{T}$ is not rectangular. This argument proves the second part of the assertion (2), as well. This completes the proof of the assertion (2) and thus Lemma 6. $\square$

PROOF OF LEMMA 5. We use Lemma 3 (1). The non-equilateral triangular tile T is either a scalene triangle (Case 1) or a non-equilateral isosceles triangle (Case 2).

Case 1. The tile T is a scalene triangle.

By Lemma 6 (1), the spherical tiling $\mathcal{T}$ by $F \geq 12$ congruent *scalene* triangles is one of the following entries of Table 2 (Ueno-Agaoka's complete classification [16, Table] of the spherical monohedral triangular tilings):

$$F_{48}, TF_{48}, F_{120}, G_{4n} \ (n \geq 3), \ I_{8n} \ (n \geq 4), \ \text{or}, \ TI_{16n+8} \ (n \geq 2).$$

The case $\mathcal{T} = I_{8n} \ (n \geq 4)$ or $TI_{16n+8} \ (n \geq 2)$ is impossible, by Table 2. Indeed, $\mathcal{T}$ has a vertex of type $4A$ and $4B + 2C$. Here (A, B, C) is the list of three inner angles of the triangular tile of $\mathcal{T}$. By $A = 1/2$ and Proposition 1, the cyclic list of four inner angles of the tile Q of $\mathcal{Q}$ is $(\alpha, \beta, \alpha, \delta) = (A, 2B, A, 2C)$. Hence, some $\mathcal{T}$ -vertex of type $4B + 2C$ causes some $\mathcal{Q}$ -vertex v of type $2\beta + \delta$. So the $\mathcal{T}$ -edge CA and the $\mathcal{T}$ -edge BA are both $\mathcal{Q}$ -edges and should match, but this is *impossible* because the triangular tile ABC of $\mathcal{T}$ is scalene.

Therefore the remaining cases are $\mathcal{T} = F_{48}, TF_{48}, F_{120}, G_{4n} \ (n \geq 3)$.

Case 1.1. $\mathcal{T} = F_{48}, F_{120}$ or TF_{48} .

If a spherical tiling $\mathcal{T} = F_{48} \ (F_{120}, \text{ resp.})$ by congruent right scalene triangles yields a spherical monohedral quadrangular tiling $\mathcal{Q}$, then $\mathcal{Q}$ is $K_{24} \ (K_{60}, \text{ resp.})$, by the construction (Subsection 2.1) of $K_{24} \ (K_{60}, \text{ resp.})$ and the following:

FACT 3. *Every spherical tiling $\mathcal{T}$ by congruent right scalene triangles T yields at most one spherical monohedral quadrangular tiling $\mathcal{Q}$, modulo congruence.*

PROOF. By Proposition 1, the tile Q of $\mathcal{Q}$ cannot be a parallelogram. If $\mathcal{T}$ yields another spherical monohedral quadrangular tiling, we must remove a cathetus of T , so some inner angle of Q should be 1. This is a contradiction. $\square$

A spherical monohedral triangular tiling TF_{48} yields gK_{24} . It is proved as follows: For each tile of gK_{24} , draw a diagonal segment between opposite

vertices having unequal inner angles. This subdivision of the tiles results in a spherical tiling $\mathcal{Q}'$ consisting of 48 congruent right scalene triangles having inner angles $1/3$ and $1/4$. It is because K_{24} has the same tile by the construction of gK_{24} . According to Table 2, $\mathcal{Q}'$ is either F_{48} or TF_{48} . F_{48} yields K_{24} but not gK_{24} by Fact 3. So TF_{48} yields gK_{24} .

Case 1.2. $\mathcal{T} = G_{4n}$ ($n \geq 3$).

Let the three inner angles A , B , C of the triangular tile T of $\mathcal{T} = G_{4n}$ satisfy that $A + B = 1$, $C = 1/n$, and $1/(2n) < A, B < (2n - 1)/(2n)$ as in Table 2. Since the triangular tile T is scalene, none of A and B is $C = 1/n$, and we can assume $A > B$, without loss of generality. Hence

$$1 - \frac{1}{2n} > 1 - B = A > \frac{1}{2} > B > \frac{1}{2n} \quad \text{and} \quad B \neq \frac{1}{n}. \quad (4)$$

The Mollweide-like projection of G_{4n} is given by the thick edges and the thin edges of Figure 5, according to [16, Figure 4, p. 471]. Let the spherical monohedral triangular tiling $\mathcal{T} = G_{4n}$ yield a tiling $\mathcal{Q}$ consisting of $2n$ congruent quadrangles Q . If Q is obtained by gluing two copies of the triangular

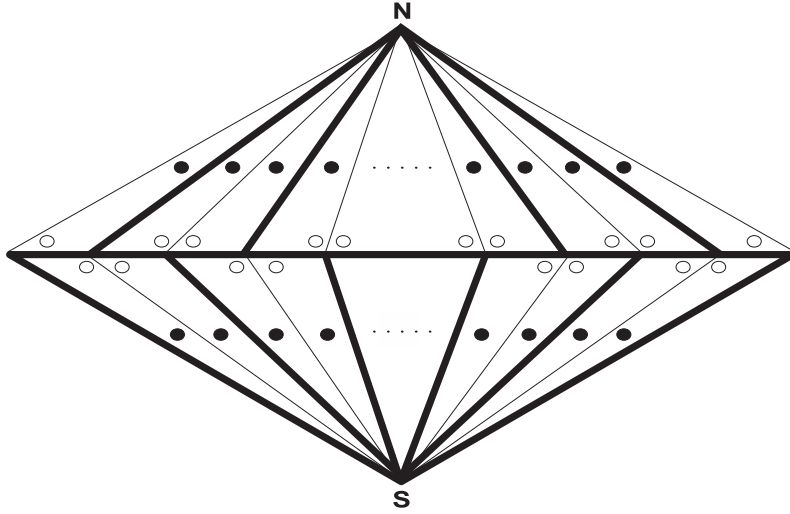


Fig. 5. The set of the thick edges and thin edges represents the Mollweide-like projection of G_{4n} , according to [16, Figure 4, p. 471]. The marks $\circ$, $\bullet$ in the figure stand for the angles A , C , while the other inner angles are B . Identify the leftmost thin (thick, resp.) edge with the rightmost thin (thick, resp.) edge. The tiling TR_{2n}^A by congruent kites is determined by the set of the thin edges (NA 's and SA 's) and the horizontal thick edges (AB 's), and the tiling TR_{2n}^B by congruent darts is determined by the thick edges (NB 's, SB 's, and AB 's). Here we suppose the inner angles satisfy $A + B = 1$, $C = 1/n$, $A > B$. See Case 1.2.

tile T of $\mathcal{T}$ at their edges AB , then the tile Q becomes a lune (i.e, a digon) but not a quadrangle because $A + B = 1$. Hence the tile Q is obtained either by gluing two copies of tile T (Case (i)) at the edges AC (thin edges in Figure 5) or (Case (ii)) at the edges BC .

Case (i). The graph of $\mathcal{Q}$ is given by the thick edges of Figure 5, and is of the form Figure 3 (right). The cyclic list of four inner angles of Q is $(B, 2A, B, 2C)$. The tile Q is a dart, because Lemma 2 (2) and the above (4). Thus the tiling $\mathcal{Q}$ is listed as TR_{2n}^B ($1/(2n) < B < 1/2, B \neq 1/n$) in the fifth entry of Table 1 with $(\alpha, \beta, \alpha, \delta) = (B, 2C, B, 2A)$.

Case (ii). The graph of $\mathcal{Q}$ is given by the thin edges of Figure 5 and the horizontal thick edges (the “equator.”) The cyclic list of four inner angles of Q is $(A, 2B, A, 2C)$. By above (4), the angle A ranges over $(1/2, 1 - 1/(2n)) \setminus \{1 - 1/n\}$. By Lemma 2 (3), the tiling $\mathcal{Q}$ is TR_{2n}^A of the form Figure 3 (left).

Applying the same argument of Case (i) and Case (ii) to the case $A < B$, we obtain TR_{2n}^A where $A \in (1/(2n), 1 - 1/(2n)) \setminus \{1/2, 1 - 1/n, 1/n\}$.

Case 2. $\mathcal{T}$ is a spherical tiling by $F \geq 12$ congruent non-equilateral isosceles triangles T .

Let B and C be the two base angles of T and A ($\neq B = C$) be the other inner angle of the triangular tile T . If a kite or a dart is obtained from gluing two copies of the isosceles triangle T , it has inner angle $2A$. Actually, the cyclic list of the four inner angles is $(2A, B, 2C, B)$, $(2A, C, 2B, C)$, or $(2A, B, C + B, C)$. Thus $\mathcal{T}$ cannot have a vertex type where A occurs odd times. So $\mathcal{T}$ is none of $F_{12}^I, F_{12}^{II}, F_{12}^{III}, F_{24}, F_{60}^I, F_{60}^{II}, H_{4n}$ ($5 \neq n \geq 3$) and TH_{8n+4} ($n \geq 3$). (In Table 2, the non-base inner angle of the non-equilateral isosceles triangular tile of H_{4n} ($5 \neq n \geq 3$) (TH_{8n+4} ($n \geq 3$)) is C .) Thus if $B = C > A$, then no tiling is possible, by Lemma 4 (II) and Lemma 6 (2). Hence, the tiling $\mathcal{T}$ amounts to be G_{4n} ($n \geq 3$) where $A = 1 - 1/n$ and $B = C = 1/n$, by the previous two sentences, Lemma 4 (I), and Lemma 6 (1), since Lemma 6 (2) implies $\mathcal{T} \neq I_{24}$.

From [16, Figure 11], we observe that the Mollweide-like projection of G_{4n} ($A = 1 - 1/n > B = C = 1/n, n \geq 3$) is given by the thick edges (NC 's, SC 's, and AC 's) and the thin edges (NA 's and SA 's) of Figure 6. Here the upper leftmost thin (thick, resp.) edge is identified with the upper rightmost thin (thick, resp.) edge. In Figure 6, the Mollweide-like projection of the spherical monohedral dart-faced tiling $\mathcal{Q}$ is determined by the thick edges (NC 's, SC 's, and AC 's). Note that the set of the thin edges (NA 's and SA 's) and the horizontal thick edges (CA 's) determines the Mollweide-like projection of a spherical monohedral *rhombus-faced* tiling by $B = C$.

The cyclic list $(\alpha, \beta, \alpha, \delta)$ of inner angles of the quadrangular tile of $\mathcal{Q}$ is $(C, 2B, C, 2A) = (1/n, 2/n, 1/n, 2 - 2/n)$. So the tiling is indeed a dart-

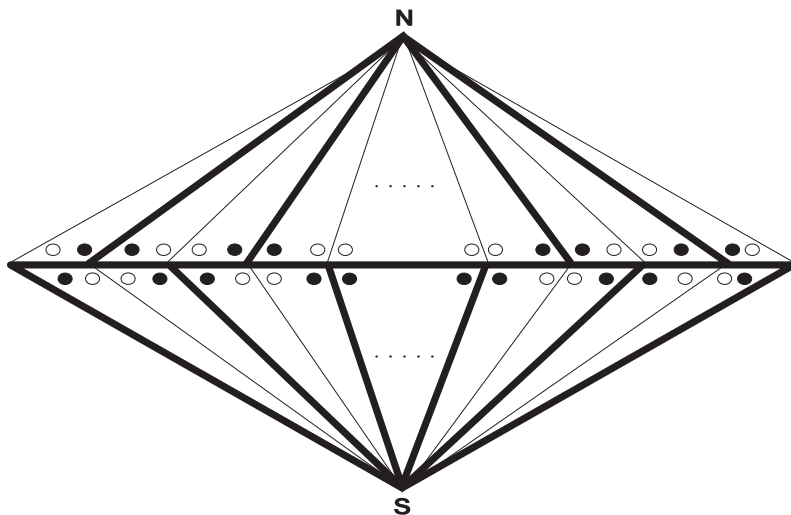


Fig. 6. The *Mollweide-like* projection of G_{4n} is given by the thick and the thin edges, and that of $TR_{2n}^{1/n}$ is given by the thick edges. The marks $\circ$, $\bullet$ stand for the inner angles A , C , while the other inner angles are B . See Case 2 of the proof of Lemma 5 and Subsubsection 3.1.2 (4).

faced $TR_{2n}^{1/n}$ by Lemma 2 (2) and $n \geq 3$. This completes the proof of Lemma 5. $\square$

By Lemma 5 and Lemma 3 (1), we have verified Theorem 2 (1) for the case the tile is a kite or a dart. $\square$

3.1.2. Theorem 2 (2). By using a procedure [5, p. 55 and Figure 3.10 (p. 56)], we will prove Theorem 2 (2) for spherical monohedral kite-faced tilings (1) K_{24} and K_{60} , (2) gK_{24} and (3) TR_{2n}^α ($1/2 < \alpha < 1 - 1/(2n)$, $\alpha \neq 1 - 1/n$, $n \geq 3$). Then we will prove Theorem 2 (2) for the spherical monohedral dart-faced tiling (4) TR_{2n}^α ($1/(2n) < \alpha < 1/2$, $n \geq 3$).

(1) K_{24} (K_{60} , resp.) is isohedral. It is proved by Lemma 1 as follows: Each triangular (pentagonal, resp.) tile Φ of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) can be transformed to another tile Φ' of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.) by the symmetry of $\mathcal{O}_8$ ($\mathcal{D}_{12}$, resp.). Within the tile Φ , any tile of K_{24} (K_{60} , resp.) can be transformed to any other tile of K_{24} (K_{60} , resp.), due to the following:

FACT 4. Let $\mathcal{P}$ be the central projection of a Platonic solid with the faces being p -gons to the sphere. Then for each tile T of $\mathcal{P}$, a p -fold axis of rotation of $\mathcal{P}$ is through the center of T .

PROOF. See [5, p. 47] for $\mathcal{P}$ being the central projection of the regular tetrahedron, [5, p. 49] for $\mathcal{P} = \mathcal{D}_{12}$ and H_{20} , and [5, p. 48] for the other $\mathcal{P}$. $\square$

Hence any tile of K_{24} (K_{60} , resp.) can be transformed to any other tile of K_{24} (K_{60} , resp.), as desired.

The cyclic list of inner angles of the tile and the list of vertex types of K_{24} (K_{60} , resp.) are as in Table 1. It is due to Fact 2, since according to Lemma 5, the graph of K_{24} (K_{60} , resp.) is obtained from that of F_{48} (F_{120} , resp.), by deleting all the hypotenuses of the triangular tiles joining the vertex B and vertex C .

(2) The list of inner angles of the tile of gK_{24} is the list of inner angles of the tile of K_{24} , by the construction (Subsection 2.2).

The Schönflies symbol of gK_{24} is D_{4d} . We prove it according to the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

The polar axis is the only four-fold rotation axis of gK_{24} . The polar axis of gK_{24} is a four-fold axis of rotation, because gK_{24} is obtained from K_{24} by gliding the northern hemisphere around the four-fold, polar axis of K_{24} . Assume ρ is another four-fold axis of rotation. Then ρ is not through a tile, as the tile is not regular. The four-fold axis ρ of rotation is not through a vertex, because every four-valent vertex other than the pole is incident to edges of different length.

gK_{24} has no three-fold axis of rotation. Assume ρ is a three-fold axis of rotation. Then ρ is through neither a quadrangular tile nor a four-valent vertex. If ρ is through a three-valent vertex, then the antipodal point is strictly between edges (or is a four-valent vertex), according to the image (Figure 2 (right)).

By the same image, the tiling gK_{24} has (i) four two-fold axes of rotation perpendicular to the four-fold axis of rotation, and (ii) four mirror planes passing between two-fold axes of rotation. Each mirror plane contains exactly six consecutive edges. Thus the Schönflies symbol of gK_{24} is D_{4d} , as desired.

The tiling gK_{24} is not isohedral, because the number 24 of the tiles does not divide the order 16 of the symmetry group D_{4d} .

The cyclic list of inner angles of the tile and the list of vertex types of gK_{24} are as in Table 1. It is due to Fact 2 and Lemma 5, because, as in Figure 2 (left), we can obtain the graph (thick edges and dotted edges) of gK_{24} from the graph (thin edges, thick edges and dotted edges) of TF_{48} by deleting each hypotenuse joining a vertex of the inner angle B and a vertex of the inner angle C ($\neq B$).

(3) In Figure 3 (left), for the cyclic list of inner angles $(\alpha, \beta, \alpha, \delta)$ of TR_{2n}^α , we see the type of each pole is $n\beta$ and the type of each non-pole vertex is $2\alpha + \delta$. Hence, the inner angles and the vertex types are as in Table 1.

The tile is a *kite*, by Lemma 2 (3) and the condition on the inner angle α . Thus the length a of each meridian edge is not equal to the length b of each non-meridian edge, in the tiling.

We prove that the Schönflies symbol of TR_{2n}^z is D_{nd} , by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If TR_{2n}^z has a three-fold or four-fold axis of rotation passing through a vertex, then the axis is the polar axis. It is verified as follows: Every non-pole vertex is three-valent and incident to an edge of length a and to two edges of length b ($\neq a$). A three-fold axis of rotation is not through an inner point of a tile T , because T is a non-regular quadrangle. Obviously, a three-fold or four-fold axis of rotation cannot pass through a point of an edge of TR_{2n}^z .

Clearly, TR_{2n}^z has an n -fold axis ρ of rotation through the poles.

TR_{2n}^z has $2n$ two-fold axes of rotation, perpendicular to ρ . It is because TR_{2n}^z has $2n$ points where the equator and the non-meridian edges intersect, and has $2n$ vertical mirror planes which pass between the horizontal two-fold axes of rotation. So the Schönflies symbol of TR_{2n}^z is D_{nd} , as desired.

TR_{2n}^z is isohedral. Indeed, if two quadrangular tiles T and T' of the tiling share a meridian edge of the tiling, then T is rotated to T' around the vertical n -fold axis of rotation. If T and T' share a non-meridian edge of the tiling TR_{2n}^z , then T is rotated to T' around a horizontal two-fold axis of rotation.

(4) It is proved as in the previous assertion (3).

This completes the proof of Theorem 2 (1) and (2) for the case where the tile is a kite or a dart. $\square$

3.2. A complete classification of spherical monohedral rhombus-faced tilings.

3.2.1. The completeness (Theorem 2 (1)).

LEMMA 7. Suppose $\mathcal{R}$ is a spherical monohedral rhombus-faced tiling and suppose that $\mathcal{T}$ is a spherical tiling by congruent non-equilateral isosceles triangles T such that the list of inner angles A, B, C of T satisfy $A > B = C$. Then if $\mathcal{T}$ yields $\mathcal{R}$, then one of the following holds:

- (1) $\mathcal{R}$ is $TR_6^{2/3}$ while $\mathcal{T}$ is $F_{12}^I, F_{12}^{II}, F_{12}^{III}, MTG_{12}^I, H_{12}, TG_{12}$ ($A = 2/3, B = C = 1/3$), or G_{12} ($A = 2/3, B = C = 1/3$);
- (2) $\mathcal{R}$ is R_{12} while $\mathcal{T}$ is F_{24} or I_{24} ;
- (3) $\mathcal{R}$ is R_{30} while $\mathcal{T}$ is F_{60}^I or F_{60}^{II} ;
- (4) $\mathcal{R}$ is $TR_8^{3/4}$ while $\mathcal{T}$ is H_{16} ($A = 1/2, B = C = 3/8$. In the entry of H_{16} of Table 2, we read $A = B = 3/8 < C = 1/2$) or $\mathcal{T}$ is G_{16} ($A = 3/4, B = C = 1/4$);
- (5) $\mathcal{R}$ is $TR_{2n}^{1-1/n}$ and $\mathcal{T}$ is G_{4n} with $A = 1 - 1/n, B = C = 1/n$ for $n \geq 5$; or
- (6) $\mathcal{R}$ is gTR_{4n+2} and $\mathcal{T}$ is TG_{8n+4} with $A = 2n/(2n+1), B = C = 1/(2n+1)$ for $n \geq 2$.

We will prove the previous lemma by using Lemma 3 (2) and the following “uniqueness” lemma:

LEMMA 8. *Every spherical tiling $\mathcal{T}$ by congruent non-equilateral isosceles triangles T yields a spherical monohedral rhombus-faced tiling $\mathcal{R}$ uniquely modulo congruence. If the list of inner angles of T is (A, B, C) and $A \neq B = C$, then the cyclic list of inner angles of the rhombic tile of $\mathcal{R}$ is $(A, 2B, A, 2C)$.*

PROOF. By removing the base edges of $\mathcal{T}$, we obtain a spherical monohedral rhombus-faced tiling. If $\mathcal{T}$ yields another spherical monohedral quadrangular tiling, then the cyclic list of inner angles of Q should be $(2B, C, 2A, C)$. Since $A \neq B$, the tile Q is a kite or a dart. $\square$

- PROPOSITION 2. (1) *For any $n \geq 3$, the spherical monohedral triangular tiling G_{4n} ($A = 1 - 1/n, B = C = 1/n$) (H_{4n} , resp.) is obtained from the spherical monohedral rhombus-faced tiling $TR_{2n}^{1-1/n}$ by drawing the meridian (the “horizontal,” resp.) diagonal segment of each tile. $I_{8n} = \text{kis}(TR_{2n}^{1-1/n})$ ($n \geq 3$).*
- (2) *The spherical monohedral triangular tiling TH_{8n+4} ($n \geq 3$) is obtained from the spherical monohedral rhombus-faced tiling gTR_{4n+2} by drawing the shorter diagonal segment of each tile. $TI_{16n+8} = \text{kis}(gTR_{4n+2})$ ($n \geq 2$).*

PROOF. (1) See Figure 11, Figure 12, Figure 14, Figure 20, and Figure 21 of [16]. (2) See [16, Figure 21]. $\square$

THE PROOF OF LEMMA 7. We use Lemma 3 (2). Let $\mathcal{T}$ be one of spherical monohedral triangular tilings listed in Lemma 4 (I).

Case 1. $\mathcal{T}$ is $F_{12}^I, F_{12}^{II}, F_{12}^{III}, MTG_{12}^I, H_{12}, G_{12}$ ($A = 2/3, B = C = 1/3$), or TG_{12} ($A = 2/3, B = C = 1/3$).

We draw one diagonal segment in each square tile of $TR_6^{2/3}$. Then the resulting tiling is exactly a tiling $\mathcal{T}$. Combinatorially, there exist just seven tilings, which are exactly the tilings $\mathcal{T}$ of this case, according to [16, pp. 481–482]. So $\mathcal{T}$ yields $TR_6^{2/3}$.

Case 2. $\mathcal{T}$ is F_{24} or I_{24} .

By the construction (Subsection 2.4) of R_{12} , the spherical tiling F_{24} by congruent right, non-equilateral isosceles triangles yields R_{12} .

The spherical tiling I_{24} by congruent non-equilateral isosceles triangles yields R_{12} , too. It is verified as follows: According to Subsection 2.4, R_{12} is constructed from $F_{24} = \text{kis}(\mathcal{O}_8)$ by deleting the 12 edges of $\mathcal{O}_8$. By recalling the image [16, Figure 20] of R_{12} and the duality between $TR_6^{2/3}$ and $\mathcal{O}_8$, we can construct R_{12} from $I_{24} = \text{kis}(TR_6^{2/3})$ by deleting the 12 edges of $TR_6^{2/3}$.

Case 3. $\mathcal{T}$ is F_{60}^I or F_{60}^{II} .

By the construction (Subsection 2.4) of R_{30} , spherical monohedral triangular tiling F_{60}^I yields R_{30} . Moreover, F_{60}^{II} yields R_{30} , by [16, Figure 36].

Case 4. $\mathcal{T}$ is H_{16} or G_{4n} ($A = 1 - 1/n, B = C = 1/n, n \geq 4$).

Proposition 2 (1) implies the assertion (4) and the assertion (5) of Lemma 7.

Case 5. $\mathcal{T}$ is TG_{8n+4} ($A = 2n/(2n+1), B = C = 1/(2n+1), n \geq 2$).

Then $\mathcal{T}$ yields gTR_{4n+2} . It is proved as follows: subdivide each rhombic tile of gTR_{4n+2} into two congruent non-equilateral isosceles triangles, by drawing the *longer* diagonal segment. This results in a spherical monohedral tilings $\mathcal{T}'$ consisting of $8n+4$ non-equilateral isosceles triangles. The cyclic list of inner angles of the rhombic tile of gTR_{4n+2} and the cyclic list of inner angles of the rhombic tile of $TR_{4n+2}^{2n/(2n+1)}$ are both $(2n/(2n+1), 2/(2n+1), 2n/(2n+1), 2/(2n+1))$, by the construction of gTR_{4n+2} . So, the cyclic list of inner angles of the isosceles triangular tile of $\mathcal{T}'$ are $(2n/(2n+1), 1/(2n+1), 1/(2n+1))$.

Owing to Ueno-Agaoka's classification (Table 2) of all spherical monohedral triangular tilings, the spherical monohedral triangular tiling $\mathcal{T}'$ is either G_{8n+4} or TG_{8n+4} . Here we have verified that G_{8n+4} yields $TR_{4n+2}^{2n/(2n+1)}$, in the previous Case 4. By Lemma 8, a spherical monohedral, isosceles triangular tiling TG_{8n+4} yields gTR_{4n+2} . This completes the proof of Lemma 7. $\square$

By Lemma 7 and Lemma 3 (2), we established Theorem 2 (1) for the case the tile is a rhombus. $\square$

3.2.2. Theorem 2 (2).

LEMMA 9. gTR_{4n+2} ($n \geq 2$) is the spherical monohedral, non-isohedral rhombus-faced tiling by the tiles of spherical monohedral tiling $TR_{4n+2}^{2n/(2n+1)}$. The smallest inner angle β and the largest inner angle α of the rhombic tile are

$$\beta = \frac{2}{2n+1}, \quad \alpha = \frac{2n}{2n+1}.$$

The Schönflies symbol of gTR_{4n+2} is D_2 . The vertex types of gTR_{4n+2} are as in Table 1.

Actually, the edges of gTR_{4n+2} have length

$$a = \arccos\left(\cos \frac{\pi\beta}{2} \middle/ \left(1 + \cos \frac{\pi\beta}{2}\right)\right) \neq \frac{\pi}{2}. \quad (5)$$

The vertices v_i, u_i ($0 \leq i \leq 2n+1$) have the following cartesian coordinates:

$$v_i = \left(\sin a \cos \frac{i\pi\beta}{2}, -\sin a \sin \frac{i\pi\beta}{2}, (-1)^i \cos a \right). \quad (6)$$

$$u_i = \left(\frac{(-1)^i - \cos \frac{i\pi\beta}{2}}{2} \sin 2a, \sin a \sin \frac{i\pi\beta}{2}, \sin^2 a \cos \frac{i\pi\beta}{2} + (-1)^i \cos^2 a \right). \quad (7)$$

PROOF. By Lemma 8 and Fact 2, the vertex types of gTR_{4n+2} are those of TG_{8n+4} applied by a substitution ($A := \alpha, B := \beta/2, C := \beta/2$).

(5) is proved by applying *spherical cosine law for angles* [17, Proposition 2.8] to the following isosceles triangle T : The rhombic tile of gTR_{4n+2} is subdivided into two copies of T such that the base edge has length $\pi - a$, the non-base inner angle is $\pi\alpha$ radian, and the two base inner angles are $\pi\beta/2$ radian.

Recall the construction (Subsection 2.5) of the tiling gTR_{4n+2} . The western hemisphere $\mathcal{W}$ of gTR_{4n+2} is that of a spherical monohedral rhombus-faced tiling $TR_{4n+2}^{2n/(2n+1)}$ (see the first and the second images of Figure 4). We set xyz -coordinate system so that the north pole N is $(0, 0, 1)$, the part of the unit sphere with positive x -coordinates is the front hemisphere, and the hemisphere $\mathcal{W}$ is the part of the unit sphere with negative y -coordinates.

The vertices of the western hemisphere $\mathcal{W}$ are: the north pole u_0 , the south pole u_{2n+1} , and v_i ($0 \leq i \leq 2n+1$) where the angle from u_0 to each v_i is $\pi/2 - (-1)^i(\pi/2 - a)$ and the longitude of each v_i from the boundary of $\mathcal{W}$ is $-i\pi\beta/2$ radian. So, the cartesian coordinates of the vertices on $\mathcal{W}$ are as in (6).

The edges of the hemisphere $\mathcal{W}$ are:

$$\{u_0, v_k\} \quad (k = 0, 2, 4, \dots, 2n), \quad (8)$$

$$\{u_{2n+1}, v_l\} \quad (l = 1, 3, 5, \dots, 2n+1), \quad (9)$$

$$\{v_j, v_{j+1}\} \quad (j = 0, 1, 2, \dots, 2n). \quad (10)$$

The eastern hemisphere $\mathcal{E}$ of gTR_{4n+2} is the image of the western hemisphere $\mathcal{W}$ by the orthogonal transformation $T_a K$. Here K is the π -radian rotation operation around the z -axis, and T_a is the rotation in a radian around the y -axis from the z -axis toward the x -axis.

So the vertices of gTR_{4n+2} on the hemisphere $\mathcal{E}$ are

$$u_i := T_a K v_i \quad (0 \leq i \leq 2n+1). \quad (11)$$

Hence the cartesian coordinate of u_i is (7). By this, u_0 and u_{2n+1} are indeed the north pole and the south pole, and thus $T_a K N = v_0$ and $T_a K S = v_{2n+1}$.

The edges of the eastern hemisphere $\mathcal{E}$ are the edges (8), (9), and (10) with u and v swapped.

This is exactly the graph of Figure 4 (right).

We will prove the angle assignment of gTR_{4n+2} is as in Figure 4 (right), as follows: If an inner angle of gTR_{4n+2} corresponds to some inner angle of the western hemisphere of $TR_{4n+2}^{2n/(2n+1)}$, then the type of the inner angle is α or β as in Figure 4 (right). The two newly arisen inner angles $\angle v_1 v_0 u_{2n}$ and $\angle u_1 u_0 v_{2n}$ of gTR_{4n+2} near the boundary of $\mathcal{W}$ have the desired values. For example, $\angle v_1 v_0 u_{2n} = \angle v_1 v_0 u_{2n+1} + \angle u_{2n+1} v_0 u_{2n} = \angle v_1 v_0 u_{2n+1} + \angle v_{2n} v_0 v_{2n+1} = \angle v_0 u_0 v_2$ and so on.

We will prove that the Schönflies symbol of gTR_{4n+2} is D_2 , by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If gTR_{4n+2} has a $k(\geq 2)$ -fold axis ρ of rotation, then $k = 2$. To see it, let p be a point where ρ and gTR_{4n+2} intersects. If p is an inner point of a tile of gTR_{4n+2} , then $k = 2$ because the tile is a rhombus. If $k > 2$, then p is not a vertex of the tiling gTR_{4n+2} , because each vertex type of the tiling is of the form $iA + mB$ for some different inner angles A, B , some positive integer m , and some positive integer $i < 3$. If p is on a edge, then $k = 2$ clearly.

gTR_{4n+2} has three mutually perpendicular two-fold axes of rotation:

- (1) The y -axis.

The π -radian rotation around it transforms the vertex v_i to the vertex v_{2n+1-i} and the vertex u_i to the vertex u_{2n+1-i} . This rotation preserves the edges of gTR_{4n+2} . Indeed, in the western hemisphere, this rotation transforms the edges (8) to the edges (9) and vice versa, and preserves the edges (10). In the eastern hemisphere, this rotation transforms the edges similarly.

- (2) ℓ through the midpoint of u_0 and v_0 .

The π -radian rotation around it is a transformation $T_{a/2}KT_{-a/2}$. The transformation maps the vertex $u_i = T_a K v_i$ ($0 \leq i \leq 2n+1$) to $(T_{a/2}K)^2 v_i = v_i$, and vice versa. Clearly, this rotation preserves the edges of gTR_{4n+2} .

- (3) ℓ' through the midpoint of u_0 and v_{2n+1} .

The π -radian rotation around it is the composition of the two π -radian rotations around the y -axis and ℓ .

To show that the tiling gTR_{4n+2} has no mirror plane passing between two of the three two-fold axes, it is sufficient to show that $v_0 u_0 v_{2n+1} u_{2n+1}$ is a proper rectangle, because the symmetry of gTR_{4n+2} fixes the proper rectangle. Here $v_0 u_0 v_{2n+1} u_{2n+1}$ is indeed a rectangle, because the vertex v_0 is antipodal to v_{2n+1} by (6) and (7) and the vertex u_0 is antipodal to u_{2n+1} . The rectangle is not regular, because the edge-length a of gTR_{4n+2} is not $\pi/2$ radian.

The xz -plane containing the two axes ℓ and ℓ' of rotation is not a mirror plane of gTR_{4n+2} . To see it, first observe that the y -coordinate of the vertex v_0 of gTR_{4n+2} is 0. Among the vertices adjacent to v_0 , the number of vertices

having positive y -coordinates is, however, greater than that of vertices having negative y -coordinates, by (7) and $\beta = 2/(2n+1)$. Therefore the Schönflies symbol of gTR_{4n+2} is D_2 .

gTR_{4n+2} ($n \geq 2$) is not isohedral, because the number $4n+2$ of tiles does not divide the order four of the symmetry group D_2 . This completes the proof of Lemma 9. $\square$

We finish the proof of Theorem 2 (2), by verifying the data of the following spherical monohedral rhombus-faced tilings given in Table 1:
(1) $TR_{2n}^{1-1/n}$ and (2) R_{12} and R_{30} .

(1) By Fact 1, the Schönflies symbol of $TR_6^{2/3}$ is O_h , and the symmetry group transitively acts on the tiling $TR_6^{2/3}$.

Consider a spherical monohedral rhombus-faced tiling $TR_{2n}^{1-1/n}$ with $n > 3$. Then, the list of three inner angles around any three-valent vertex of $TR_{2n}^{1-1/n}$ is $(\alpha, \alpha, \delta) = (1 - 1/n, 1 - 1/n, 2/n)$, where $\alpha \neq \delta$ because $n > 3$. Hence, we can prove that the Schönflies symbol of $TR_{2n}^{1-1/n}$ is D_{nd} and the tiling $TR_{2n}^{1-1/n}$ is isohedral, as in the proof for the spherical monohedral kite-faced tiling TR_{2n}^α ($1/2 < \alpha < 1 - 1/(2n)$, $\alpha \neq 1 - 1/n$, $n \geq 3$) of Theorem 2 (2).

(2) The Schönflies symbol of the spherical tiling F_{24} (F_{60}^I , resp.) by congruent isosceles triangles is O_h (I_h , resp.), because of [5, p. 50] and the following:

LEMMA 10. *If $\mathcal{P}$ is the central projection of some Platonic solid to the sphere, then every symmetry operation of $\mathcal{P}$ is that of $\text{kis}(\mathcal{P})$, and $\text{kis}(\mathcal{P})$ is isohedral.*

PROOF. Every symmetry operation of $\mathcal{P}$ is that of $\text{kis}(\mathcal{P})$, because it transforms a tile of $\mathcal{P}$ to a tile of $\mathcal{P}$, a vertex of $\mathcal{P}$ to a vertex of $\mathcal{P}$, and the center of a tile of $\mathcal{P}$ to the center of a tile of $\mathcal{P}$.

Suppose a tile T of $\mathcal{P}$ is a regular p -gon. By Fact 4, within the tile T , the p tiles of $\text{kis}(\mathcal{P})$ rotated to each other around a p -fold axis. Moreover T is transformed to another tile of $\mathcal{P}$, by a symmetry operation of $\mathcal{P}$. Thus, in $\text{kis}(\mathcal{P})$, any tile is transformed to any other tile by a symmetry operation of $\text{kis}(\mathcal{P})$. $\square$

The Schönflies symbol of R_{12} (R_{30} , resp.) is that of F_{24} (F_{60}^I , resp.), because each symmetry operation of F_{24} (F_{60}^I , resp.) preserves the tiles and the edges of F_{24} (F_{60}^I , resp.).

By the construction (Subsection 2.4) of R_{12} (R_{30} , resp.) and Fact 4, if T , T' are tiles of R_{12} (R_{30} , resp.) sharing the center c of a tile of $\mathcal{O}_8$ (H_{20} , resp.), then T is rotated to T' around the three-fold (five-fold, resp.) axis ρ of the rotation such that ρ passes through the center c . If T and T' does not share any vertex, then some symmetry operation of $\mathcal{O}_8$ (H_{20} , resp.) rotates T to

another tile T'' of R_{12} (R_{30} , resp.) such that T'' and T' share the center of some tile of $\mathcal{O}_8$ (H_{20} , resp.).

By Table 2, the list of inner angles of the isosceles triangular tile T of F_{24} (F_{60}^I , resp.) is $(A, B, C) = (2/3, 1/4, 1/4)$ ($(2/3, 1/5, 1/5)$, resp.). As we glue two copies of T at the base edges, the cyclic list of inner angles of the rhombic tile of R_{12} (R_{30} , resp.) is $(\alpha, \beta, \alpha, \beta) = (2B, A, 2B, A)$. So, the vertex types of R_{12} (R_{30} , resp.) are as in Table 1, by Fact 2. $\square$

4. The anisohedral spherical triangles

We will prove that the tiles of the following spherical monohedral triangular tilings are *not* anisohedral.

LEMMA 11. *The followings are spherical isohedral triangular tilings:*

- (1) *Every instance of a continuously deformable tiling F_4 , F_{12}^I , F_{24} , F_{48} , F_{60}^I , F_{60}^{II} , and F_{120} .*
- (2) *Every instance of a continuously deformable tiling G_{4n} ($n \geq 2$), and G_{4n+2} ($n \geq 1$).*
- (3) *An instance H_{20} of an infinite series H_{4n} ($n \geq 3$).*
- (4) *An instance I_{24} of an infinite series I_{8n} ($n \geq 3$).*

PROOF. (1) Let $\mathcal{T}$ be an instance of F_4 . $\mathcal{T}$ has two two-fold axes of rotation, each of which is through the midpoint of an edge and that of the opposite edge, in view of the development map of F_4 [16, Figure 17]. We can transform any tile T of $\mathcal{T}$ to any other tile T' , by π -rotation around an axis ρ which is through the midpoint of the edge shared by the two tiles T and T' . Hence F_4 is isohedral.

Next, we prove that F_{48} is isohedral. Indeed, every symmetry operation of $\mathcal{O}_8$ is that of F_{48} , as in the proof of Lemma 1.

For each tile T of $\mathcal{O}_8$ and each vertex A of T , there is a mirror plane σ passing through A and bisecting the edge of T opposite to A , because of [5, (ix) of p. 48]. Let M be the midpoint of the edge. The mirror plane σ passes through M and the center c of T .

Within the tile T of $\mathcal{O}_8$, a pair of right scalene triangular tiles of F_{48} adjacent at the edge cM are reflected to each other by the mirror plane σ , and another pair of right scalene triangular tiles of F_{48} adjacent at their hypotenuses cA are reflected to each other by another mirror plane σ . Therefore within the tile T of $\mathcal{O}_8$, the six right scalene triangular tiles of F_{48} are transformed to each other by repeated reflection operations with appropriate mirror planes σ 's of $\mathcal{O}_8$ where all such σ 's pass through the center c of T .

Consider that tiles S and S' of F_{48} are in different tiles T and T' of $\mathcal{O}_8$. Then, to S apply a symmetry operation ρ that maps T to T' , and then to the resulting tile of F_{48} apply a symmetry operation mentioned above in order to get S' . Hence, F_{48} is isohedral.

The proof of the isohedrality of F_{120} is obtained from the proof of the isohedrality of F_{48} , by replacing F_{48} with F_{120} , $\mathcal{O}_8$ with H_{20} , and “(ix) of p. 48” with “(v) and (vi) of p. 49.”

The isohedrality of F_{12}^I , F_{24}^I , F_{60}^I , and F_{60}^{II} is by Lemma 10 and their constructions (cf. $F_{60}^{II} = \text{kis}(\mathcal{D}_{12})$ by [16, Figure 17 and p. 484]).

(2) An instance $\mathcal{T}$ of G_{4n} ($n \geq 2$) has n horizontal two-fold axes of rotation and n vertical mirror planes passing between two horizontal two-fold axes, according to [16, Figure 4]. Let T and T' be any tiles of $\mathcal{T}$. If T' shares a meridian edge e of T , then T is reflected to T' by a vertical mirror plane σ of $\mathcal{T}$ such that σ contains e . Moreover, if T' shares a non-meridian edge e' of T , then there is a horizontal two-fold axis ρ of rotation of $\mathcal{T}$ such that ρ passes through the midpoint of e' and T is rotated to T' around ρ . Hence, any tile of $\mathcal{T}$ is transformed to any other tile of $\mathcal{T}$ by a symmetry operation of $\mathcal{T}$. Thus, $\mathcal{T}$ is isohedral.

G_{4n+2} ($n \geq 1$) is isohedral by the vertical $(2n+1)$ -fold axis of rotation and the horizontal mirror plane, according to [16, Figure 4].

(3) An instance H_{20} is the central projection of the regular icosahedron on the sphere, and thus the Schönflies symbol of H_{20} is I_h , and H_{20} is isohedral.

(4) This is due to Proposition 2 (1) and Lemma 10. $\square$

The other spherical triangular tilings are not isohedral.

LEMMA 12. *Spherical monohedral triangular tilings H_{4n} ($5 \neq n \geq 3$), I_{8n} ($n \geq 4$), TH_{8m+4} ($m \geq 3$), and TI_{16m+8} ($m \geq 2$) are non-isohedral.*

PROOF. For $\mathcal{T} = H_{4n}$ ($5 \neq n \geq 3$) or I_{8n} ($n \geq 4$), the Schönflies symbol of $\mathcal{T}$ is D_{nd} . It is due to Proposition 2, because every symmetry operation of $TR_{2n}^{1-1/n}$ preserves the diagonal segments of the tile and the Schönflies symbol of $TR_{2n}^{1-1/n}$ is D_{nd} by Theorem 2.

I_{8n} ($n \geq 4$) is not isohedral, since the order of the symmetry group D_{nd} of I_{8n} is $4n$ according to [5, p. 44] but the number $8n$ of the tiles of I_{8n} does not divide $4n$.

H_{4n} ($5 \neq n \geq 3$) is not isohedral, because no symmetry operation transforms a tile around the pole to a tile on the equator (See the first line of Figure 7).

We can prove that the Schönflies symbol of TH_{8m+4} ($m \geq 3$) (TI_{16m+8} ($m \geq 2$), resp.) is D_2 , in a similar manner as we proved that the Schönflies

symbol of gTR_{4n+2} is D_2 in Lemma 9, in view of the construction (Proposition 2 (2)) of TH_{8m+4} (TI_{16m+8} , resp.).

TH_{8m+4} ($m \geq 3$) (TI_{16m+8} ($m \geq 2$), resp.) is not isohedral, because the number $8m+4$ ($m \geq 3$) ($16m+8$ ($m \geq 2$), resp.) of tiles does not divide the order 4 of the symmetry group D_2 . $\square$

Although H_{12} is not isohedral, the tile of H_{12} is congruent to the tile of the tiling G_{12} with $A = 2/3, B = C = 1/3$. G_{12} is isohedral by Lemma 11 (2). The other spherical monohedral, *non-isohedral* triangular tilings listed in the previous lemma are spherical tilings by anisohedral triangles, owing to the following lemma:

- LEMMA 13. (1) *If the tile of H_{4n} ($5 \neq n \geq 4$) is the tile of another spherical monohedral triangular tiling $\mathcal{T}$, then $\mathcal{T}$ is TH_{8m+4} for some $m \geq 3$ with $4n = 8m + 4$.*
- (2) *If the tile of TH_{8m+4} ($m \geq 3$) is the tile of another spherical monohedral triangular tiling $\mathcal{T}$, then $\mathcal{T}$ is H_{8m+4} .*
- (3) *If the tile of I_{8n} ($n \geq 4$) is the tile of another spherical monohedral triangular tiling $\mathcal{T}$, then $\mathcal{T}$ is TI_{16m+8} for some $m \geq 2$ with $8n = 16m + 8$.*
- (4) *If the tile of TI_{16m+8} ($m \geq 2$) is the tile of another spherical monohedral triangular tiling $\mathcal{T}$, then $\mathcal{T}$ is I_{16m+8} .*

PROOF. (1) The number of tile of $\mathcal{T}$ is $4n$ ($5 \neq n \geq 4$) and the tile of H_{4n} is an *isosceles* triangle. So, if $\mathcal{T}$ is sporadic, then $n \geq 6$ and $\mathcal{T}$ is F_{24} , F_{60}^I or F_{60}^{II} . However, the non-base inner angle of $\mathcal{T}$ is greater than the non-base inner angle $2/n$ of H_{4n} for each case. Thus $\mathcal{T}$ is a non-sporadic spherical tiling by congruent *isosceles* triangles. Hence, $\mathcal{T}$ is G_{4n} , TG_{4n} , or TH_{8m+4} for some $m \geq 3$ such that $8m+4 = 4n$. By [16, Table] (included as Table 2), the tile of G_{4n} and that of TG_{4n} have an inner angle $1/n$ but the tile of H_{4n} does not, as $5 \neq n \geq 4$. Thus $\mathcal{T}$ is TH_{8m+4} , as desired.

(3) The number of tile of $\mathcal{T}$ is $8n \geq 32$ and the tile of I_{8n} is a right scalene triangle. So, if $\mathcal{T}$ is sporadic, then (1) $n = 6$ and $\mathcal{T}$ is F_{48} or TF_{48} , or (2) $n = 15$ and $\mathcal{T} = F_{120}$. Then the smallest inner angle of $\mathcal{T}$ is greater than the smallest inner angle $1/n$ of I_{8n} . Thus $\mathcal{T}$ is a non-sporadic spherical tiling by congruent right scalene triangles. So $\mathcal{T}$ is G_{8n} , or TI_{8n} . By [16, Table] (included as Table 2), an inner angle $1/(2n)$ of the tile of G_{8n} is smaller than the smallest angle $1/n$ of the tile of I_{8n} for each case. So the tile of $\mathcal{T}$ is not congruent to the tile of G_{8n} . Thus $\mathcal{T} = TI_{8n}$, as desired.

Assertion (2) (assertion (4), resp.) follows from assertion (1) (assertion (3), resp.), because the tile of TH_{8m+4} ($m \geq 3$) (TI_{16m+8} ($m \geq 2$), resp.) is also the tile of H_{8m+4} (I_{16m+8} , resp.). $\square$

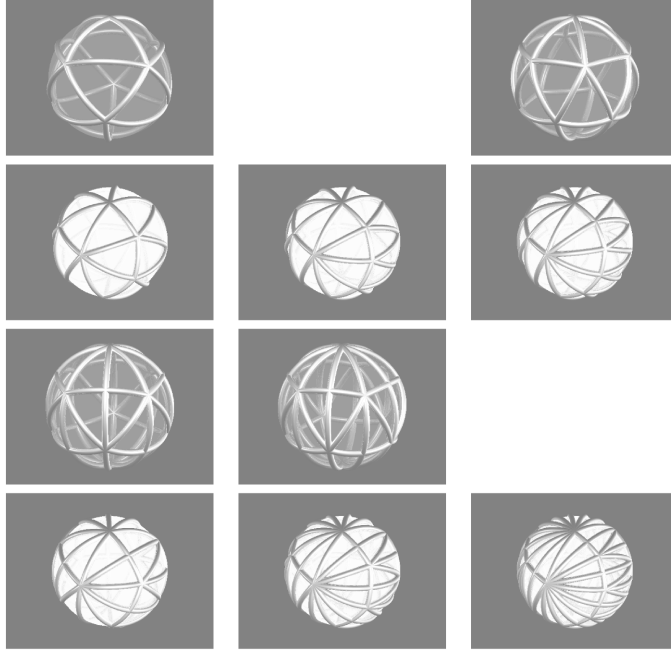


Fig. 7. The first (the second, resp.) line consists of spherical tilings H_{4n} ($n = 4, 6$) (TH_{8m+4} ($m = 3, 4, 5$), resp.) by $4n$ ($8m + 4$, resp.) congruent anisohedral isosceles triangles AI_n (AI_{2m+1} , resp.). The third (the fourth, resp.) line consists of spherical tilings I_{8n} ($n = 4, 5$) (TI_{16m+8} ($m = 2, 3, 4$), resp.) by $8n$ ($16m + 8$, resp.) congruent anisohedral right scalene triangles ARS_n (ARS_{2m+1} , resp.). See Theorem 1 and [16, Table] (included as Table 2 in this paper).

LEMMA 14. *The spherical tilings by congruent anisohedral triangles are exactly H_{4n} ($5 \neq n \geq 4$), TH_{8m+4} ($m \geq 3$), I_{8n} ($n \geq 4$), and TI_{16m+8} ($m \geq 2$).*

PROOF. H_{4n} ($5 \neq n \geq 4$), I_{8n} ($n \geq 4$), TH_{8m+4} ($m \geq 3$) and TI_{16m+8} ($m \geq 2$) are spherical tilings by congruent anisohedral triangles, because of Lemma 12 and Lemma 13.

We will prove that for any *other* $\mathcal{T}$ listed in Ueno-Agaoka's complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings but not listed in Lemma 11, the tile T of $\mathcal{T}$ is congruent to some spherical isohedral triangular tiling. We will argue such $\mathcal{T}$ from top to bottom of Table 2. If $\mathcal{T}$ is F_{12}^{II} , or F_{12}^{III} , then the tile T of $\mathcal{T}$ admits F_{12}^I , because the list of three inner angles of the tile T is the list of three inner angles of the tile of F_{12}^I . As F_{12}^I is isohedral by Lemma 11 (1), the tile of $\mathcal{T}$ is not anisohedral. The tile of TF_{48} admits F_{48} , because the list of three inner angles of T is the list of three inner angles of the tile of F_{48} . F_{48} is isohedral by Lemma 11 (1). Hence, the tile of TF_{48} is not anisohedral. If $\mathcal{T}$ is TG_{8n}

$(n \geq 2)$, TG_{8n+4} ($n \geq 1$), MTG_{8n+4}^I ($n \geq 1$), MTG_{8n+4}^{II} ($n \geq 2$), or H_{12} , then the tile T of $\mathcal{T}$ admits an instance of G_m where m is the number of tiles of $\mathcal{T}$. $\mathcal{T}$ is isohedral by Lemma 11 (2). Hence, the tile of $\mathcal{T}$ is not anisohedral. $\square$

Because Ueno-Agaoka's complete classification of spherical monohedral triangular tilings describes the inner angles of the tiles, Theorem 1 follows from the previous Lemma (see Figure 7).

THEOREM 3. *No spherical (kite/dart/rhombus) is anisohedral.*

PROOF. By Theorem 2 (2) and Table 1, the tiles of the only spherical monohedral, non-isohedral (kite/dart/rhombus)-faced tilings gK_{24} and gTR_{4n+2} ($n \geq 2$) are those of spherical *isohedral* rhombus-faced tilings K_{24} and $TR_{4n+2}^{2n/(2n+1)}$. $\square$

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WAFOM over abelian groups for quasi-Monte Carlo point sets

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ABSTRACT. In this paper, we study quasi-Monte Carlo (QMC) rules for numerical integration. J. Dick proved a Koksma-Hlawka type inequality for α -smooth integrands and gave an explicit construction of QMC rules achieving the optimal rate of convergence in that function class. From this inequality, Matsumoto, Saito and Matoba introduced the Walsh figure of merit (WAFOM) $WF(P)$ for an $\mathbf{F}_2$ -digital net P as a quickly computable quality criterion for P as a QMC point set. The key ingredient for obtaining WAFOM is the Dick weight, a generalization of the Hamming weight and the Niederreiter-Rosenbloom-Tsfasman (NRT) weight.

We extend the notions of the Dick weight and WAFOM over a general finite abelian group G , and show that this version of WAFOM satisfies Koksma-Hlawka type inequality when G is cyclic. We give a MacWilliams-type identity on weight enumerator polynomials for the Dick weight, by which we can compute the minimum Dick weight as well as WAFOM. We give a lower bound on WAFOM of order $N^{-C'_G(\log N)/s}$ and an upper bound on lowest WAFOM of order $N^{-C_G(\log N)/s}$ for given (G, N, s) if $(\log N)/s$ is sufficiently large, where C'_G and C_G are constants depending only on the cardinality of G and N is the cardinality of quadrature rules in $[0, 1)^s$. These bounds generalize the bounds given by Yoshiki and others given for $G = \mathbf{F}_2$.

1. Introduction

Quasi-Monte Carlo (QMC) integration is a method for numerical integration using the average of function evaluations as an approximation of the true integration value. In QMC integration, sample points are chosen deterministically, while in Monte-Carlo integration they are chosen randomly. Thus, how to construct point sets has been a major concern in QMC theory. One of the known good construction frameworks is digital nets, which is based on linear algebra over finite fields (or more generally over finite rings).

A strong analogy between coding theory and QMC point sets is well known (see, e.g., [2, 13, 17]). In coding theory, the minimum Hamming weight is used for a criterion for linear codes. Analogically, Niederreiter-

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Rosenbloom-Tsfasman (NRT) weight is a criterion for digital nets in QMC theory [12, 15]. More precisely, the minimum NRT weight is essentially equivalent to t -value and gives an upper bound on the star-discrepancy, which are important criteria for QMC point sets. Recently, based on Dick's work [3], Matsumoto, Saito and Matoba defined the Dick weight μ on digital nets over $\mathbf{F}_2$ and related it to a criterion called the Walsh figure of merit (WAFOM) in [10]. In this paper, as a generalization of [10], we extend the notions of the Dick weight and WAFOM for digital nets over $\mathbf{Z}_b$, and more generally, for subgroups of $G^{s \times n}$ where G is a finite abelian group. Furthermore, we establish a MacWilliams-type identity for the Dick weight, which gives a computable formula of the minimum Dick weight and WAFOM.

Let us recall the notion of QMC integration. For an integrable function $f : [0, 1]^s \rightarrow \mathbf{R}$ and a finite point set in an s -dimensional unit cube $\mathcal{P} \subset [0, 1]^s$, quasi Monte-Carlo (QMC) integration of f by $\mathcal{P}$ is an approximation value

$$I_{\mathcal{P}}(f) := \frac{1}{N} \sum_{\mathbf{x} \in \mathcal{P}} f(\mathbf{x})$$

of the actual integration

$$I(f) := \int_{[0, 1]^s} f(\mathbf{x}) d\mathbf{x},$$

where $N := |\mathcal{P}|$ is the cardinality of $\mathcal{P}$. The QMC integration error is defined as $\text{Err}(f; \mathcal{P}) := |I_{\mathcal{P}}(f) - I(f)|$. If the integrand f has bounded variation in the sense of Hardy and Krause, the Koksma-Hlawka inequality shows that $\text{Err}(f; \mathcal{P}) \leq V(f)D(\mathcal{P})$, where $V(f)$ is the total variation of f and $D(\mathcal{P})$ is the star-discrepancy of $\mathcal{P}$. There have been many studies on the construction of low-discrepancy point sets, i.e., point sets with $D(\mathcal{P}) \in O(N^{-1+\varepsilon})$. In particular, digital nets and sequences are a general framework for the construction of good point sets. We refer to [6] and [13] for the general information on QMC integration and digital nets and sequences.

Recently, higher order convergence results for digital nets, i.e., $\text{Err}(f; \mathcal{P})$ converges faster than N^{-1} , has been established. For a given integer $\alpha > 1$, Dick gave quadrature rules for α -smooth integrands which achieve $\text{Err}(f; \mathcal{P}) \in O(N^{-\alpha+\varepsilon})$ [3]. He introduced a weight which gives a bound on a criterion $\text{WF}_{\alpha}(\mathcal{P})$ (he did not give a name and we use the name in [10]) for a digital net $\mathcal{P}$ over a finite field with cardinality b , and proved a Koksma-Hlawka type inequality $\text{Err}(f; \mathcal{P}) \leq C_{b,s,\alpha} \|f\|_{\alpha} \cdot \text{WF}_{\alpha}(\mathcal{P})$, where $\|f\|_{\alpha}$ is a norm of f for a Sobolev space and $C_{b,s,\alpha}$ is a constant depend only on b , s , and α . Later he improved the constant factor of the lowest WF_{α} for digital nets over a finite cyclic group [4].

As a discretized version of Dick's method, Matsumoto, Saito and Matoba introduced the Dick weight μ and a related criterion WAFOM $\text{WF}(P)$ for an $\mathbf{F}_2$ -digital net P [10]. WAFOM also satisfies a Koksma-Hlawka type inequality (with some errors due to discretization). One remarkable merit of WAFOM is that WAFOM is easily computable by the inversion formula [10, (4.2)], which is easier to implement than the formula of WF_α derived from [1, Section 4]. Using this merit, they executed a random search of low-WAFOM point sets and showed that such point sets perform better than some standard low-discrepancy point sets. There are several studies on low-WAFOM point sets. The existence of low-WAFOM point sets was shown by Matsumoto and Yoshiki [11]. The author proved that the interlacing construction for higher order QMC point sets with Niederreiter-Xing sequences over a finite field gives low-WAFOM point sets [18].

In this paper, as a generalization of [10] we propose the Dick weight and WAFOM for digital nets over $\mathbf{Z}_b$ and for subgroups of $G^{s \times n}$ where G is a finite abelian group. WAFOM over $\mathbf{Z}_b$ is also a discretized version of Dick's method and thus satisfies a Koksma-Hlawka type inequality. Moreover, we give a MacWilliams-type identity of weight enumerator polynomials for the Dick weight. Using this identity we obtain a computable formula of the minimum Dick weight as well as WAFOM, which is a generalization of the inversion formula for WAFOM in the dyadic case. Furthermore, we give generalizations of known properties of WAFOM over $\mathbf{F}_2$ in [11] and [19]. More precisely, we give a lower bound on WAFOM and prove the existence of low-WAFOM point sets. In particular, we improve some of the results in [11]. These results imply that there exist positive constants C , D , D' and F depending only on b and independent of s , n and N such that $N^{-C \log N/s} \leq \min\{\text{WF}(P) \mid P \text{ is a digital net, } |P| \leq N\} \leq FN^{-D(\log N)/s+D'}$, if $(\log N)/s$ is sufficiently large.

These results are similar to the works of Dick, but there is no implication between them. Dick fixed the smoothness α , while our method requires n -smoothness on the function where n is as above. Thus, in our case, the function class is getting smaller for n being increased.

The rest of the paper is organized as follows. In Section 2, we introduce the necessary background and notation, such as the discretization scheme of QMC integration, the discrete Fourier transform, and Walsh functions. In Section 3, we define the Dick weight and WAFOM over a general finite abelian group G , and prove a Koksma-Hlawka type inequality in the case that G is cyclic. In Section 4, we define the weight enumerator polynomial, give the MacWilliams-type identity for the Dick weight, and give a computable formula of WAFOM. In Section 5, we give a lower bound on WAFOM, prove the existence of low-WAFOM point sets, and study the order of WAFOM.

2. Preliminaries

Throughout this paper, we use the following notation. Let $\mathbf{N}$ be the set of positive integers and $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$. Let b be an integer greater than 1. Let $\mathbf{Z}_b = \mathbf{Z}/b\mathbf{Z}$ be the residue class ring modulo b . We identify $\mathbf{Z}_b$ with the set $\{0, 1, \dots, b-1\} \subset \mathbf{Z}$. For a set S , we denote by $|S|$ the cardinality of S . For a group or a ring R and positive integers s and n , we denote by $R^{s \times n}$ the set of $s \times n$ matrices with components in R . We denote by O the zero matrix. We denote by e the base of the natural logarithm.

2.1. Discretized QMC in base b . In this subsection, we explain discretized QMC in base b . This discretization is a straightforward generalization of the $b = 2$ case in [10].

Let s be a positive integer. Let $\mathcal{P} \subset [0, 1)^s$ be a point set in an s -dimensional unit cube with finite cardinality $|\mathcal{P}| = N$, and let $f : [0, 1)^s \rightarrow \mathbf{R}$ be an integrable function. Recall that quasi-Monte Carlo integration by $\mathcal{P}$ is an approximation value

$$I_{\mathcal{P}}(f) := \frac{1}{N} \sum_{\mathbf{x} \in \mathcal{P}} f(\mathbf{x})$$

of the actual integration

$$I(f) := \int_{[0, 1)^s} f(\mathbf{x}) d\mathbf{x}.$$

The QMC integration error is $\text{Err}(f; \mathcal{P}) := |I_{\mathcal{P}}(f) - I(f)|$.

Here, we fix a positive integer n , which is called the degree of discretization or the precision. We consider an n -digit discrete approximation in base b . We associate a matrix $B := (b_{i,j}) \in \mathbf{Z}_b^{s \times n}$ with a point $\mathbf{x}_B = (x_B^1, \dots, x_B^s) = (\sum_{j=1}^n b_{1,j} b^{-j}, \dots, \sum_{j=1}^n b_{s,j} b^{-j}) \in [0, 1)^s$, and with an s -dimensional cube $\mathbf{I}_B := \prod_{i=1}^s I_i \subset [0, 1)^s$, where each edge $I_i := [x_B^i, x_B^i + b^{-n})$ is a half-open interval with length b^{-n} . We define n -digit discrete approximation f_n of f as

$$f_n : \mathbf{Z}_b^{s \times n} \rightarrow \mathbf{R}, \quad B := (b_{i,j}) \mapsto \frac{1}{\text{Vol}(\mathbf{I}_B)} \int_{\mathbf{I}_B} f(\mathbf{x}) d\mathbf{x}.$$

Let P be a subset of $\mathbf{Z}_b^{s \times n}$. We define n -th discretized QMC integration of f by P as

$$I_{P,n}(f) := \frac{1}{|P|} \sum_{B \in P} f_n(B)$$

and define the n -th discretized QMC integration error as

$$\text{Err}(f; P, n) := |I_{P,n}(f) - I(f)|.$$

For each $B \in P$, we take the center point of the cube I_B . Let $\mathcal{P} \subset [0, 1]^s$ be the set of such center points given by P . By a slight extension of [10, Lemma 2.1], if f is continuous with Lipschitz constant K then we have $|I_{P,n}(f) - I_{\mathcal{P}}(f)| \leq K\sqrt{s}b^{-n}$. We take n large enough so that $K\sqrt{s}b^{-n}$ is negligibly small compared to the order of QMC integration error $|I_{\mathcal{P}}(f) - I(f)|$ by $\mathcal{P}$. Then we may regard the n -th discretized QMC integration error $\text{Err}(f; P, n)$ as an approximation of the QMC integration error $\text{Err}(f; P)$.

As point sets, in this paper we consider subgroups of $\mathbf{Z}_b^{s \times n}$ as well as digital nets. The definition of digital nets over finite rings is given in [7], we adopt an equivalent definition of digital nets, which is proposed as digital nets with generating matrices in [5, Definition 4.3].

DEFINITION 1. Let $C_1, \dots, C_s \in \mathbf{Z}_b^{n \times d}$ be matrices and let $X_1, \dots, X_d \in \mathbf{Z}_b^{s \times n}$ be defined by the j -th row of X_i is the transpose of the i -th column of C_j . Assume that $X_1, \dots, X_d$ are a free basis of $\mathbf{Z}_b^{s \times n}$ as a $\mathbf{Z}_b$ -module. For an integer k with $0 \leq k \leq b^d - 1$, we define a matrix $\mathbf{x}_k \in \mathbf{Z}_b^{s \times n}$ as $\mathbf{x}_k = \sum_{i=1}^d \kappa_{i-1} X_i$, where $k = \kappa_0 + \kappa_1 b^1 + \dots + \kappa_{d-1} b^{d-1}$ ($0 \leq \kappa_i \leq b-1$) is the b -adic expansion of k . We call the set $\{\mathbf{x}_0, \dots, \mathbf{x}_{b^d-1}\}$ the digital net generated by the matrices $C_1, \dots, C_s$.

It is easy to see that digital nets become subgroups of $\mathbf{Z}_b^{s \times n}$.

2.2. Discrete Fourier transform. In this subsection, we recall the notion of character groups and the discrete Fourier transform. We refer to [16] for general information on character groups. Let G be a finite abelian group. Let $T := \{z \in \mathbf{C} \mid |z| = 1\}$ be the multiplicative group of complex numbers of absolute value one. Let $\omega_b = \exp(2\pi\sqrt{-1}/b)$.

DEFINITION 2. We define the character group of G by $G^\vee := \text{Hom}(G, T)$, namely $G^\vee$ is the set of group homomorphisms from G to T .

There is a natural pairing $\bullet : G^\vee \times G \rightarrow T$, $(h, g) \mapsto h \bullet g := h(g)$.

We can see that $\mathbf{Z}_b^\vee$ is isomorphic to $\mathbf{Z}_b$ as an abstract group. Throughout this paper, we identify $\mathbf{Z}_b^\vee$ with $\mathbf{Z}_b$ through a pairing $\bullet : \mathbf{Z}_b \times \mathbf{Z}_b \rightarrow T$, $(h, g) \mapsto h \bullet g := \omega_b^{hg}$, where hg is the product in $\mathbf{Z}_b$.

Let R be a commutative ring containing $\mathbf{C}$. Let $f : G \rightarrow R$ be a function. We define the discrete Fourier transform of f as below.

DEFINITION 3. The discrete Fourier transform of f is defined by $\hat{f} : G^\vee \rightarrow R$, $h \mapsto \frac{1}{|G|} \sum_{g \in G} f(g)(h \bullet g)$. Each value $\hat{f}(h)$ is called a discrete Fourier coefficient.

We assume that $P \subset G$ is a subgroup. We define $P^\perp := \{h \in G^\vee \mid h \bullet g = 1 \text{ for all } g \in P\}$. Since $P^\perp$ is the kernel of the restriction map $G^\vee \rightarrow P^\vee$, we have $|P^\perp| = |G|/|P|$. We recall the orthogonality of characters.

LEMMA 1. *Suppose that $P \subset G$ is a subgroup and $g \in G$. Then we have*

$$\sum_{h \in P^\perp} h \bullet g = \begin{cases} |P^\perp| & \text{if } g \in P, \\ 0 & \text{if } g \notin P. \end{cases}$$

This lemma implies the Poisson summation formula and the Fourier inversion formula.

THEOREM 1 (Poisson summation formula).

$$\frac{1}{|P|} \sum_{g \in P} f(g) = \sum_{h \in P^\perp} \hat{f}(h).$$

PROOF.

$$\begin{aligned} \sum_{h \in P^\perp} \hat{f}(h) &= \sum_{h \in P^\perp} \frac{1}{|G|} \sum_{g \in G} f(g) (h \bullet g) \\ &= \sum_{g \in G} \frac{1}{|G|} f(g) \sum_{h \in P^\perp} h \bullet g \\ &= \frac{1}{|G|} \sum_{g \in P} f(g) \cdot |P^\perp| \quad (\because \text{Lemma 1}) \\ &= \frac{1}{|P|} \sum_{g \in P} f(g). \end{aligned} \quad \square$$

THEOREM 2 (Fourier inversion formula). *For a complex-valued function $f : G \rightarrow \mathbb{C}$, we have $f(g) = \sum_{h \in G^\vee} \hat{f}(-h)(h \bullet g)$ for any $g \in G$. Moreover, if f is real-valued, we have $f(g) = \sum_{h \in G^\vee} \hat{f}(h)(h \bullet g)$.*

PROOF. By Lemma 1, we have $\sum_{h \in G^\vee} h \bullet g = 0$ if $g \neq 0$ and $\sum_{h \in G^\vee} h \bullet g = |G|$ if $g = 0$. Thus we have

$$\begin{aligned} \sum_{h \in G^\vee} \hat{f}(-h)(h \bullet g) &= \sum_{h \in G^\vee} \frac{1}{|G|} \sum_{g' \in G} f(g')((-h) \bullet g')(h \bullet g) \\ &= \frac{1}{|G|} \sum_{g' \in G} f(g') \sum_{h \in G^\vee} (h \bullet (g - g')) \\ &= f(g), \end{aligned}$$

which proves the complex-valued case. If f is real-valued, we have $\hat{f}(-h) = \overline{\hat{f}(h)}$, and thus the complex-valued case implies the real-valued case. $\square$

2.3. Walsh functions. In this subsection, we recall the notion of Walsh functions and Walsh coefficients, and see the relationship between Walsh coefficients and discrete Fourier coefficients. As a corollary, we prove that the n -digit discrete approximation f_n of f is essentially equal to the appropriate approximation of the Walsh series of f . We refer to [6, Appendix A] for general information on Walsh functions.

First, we define Walsh functions for the one dimensional case.

DEFINITION 4. Let $k \in \mathbf{N}_0$ with b -adic expansion $k = \kappa_0 + \kappa_1 b^1 + \kappa_2 b^2 + \cdots$ (this expansion is actually finite), where $\kappa_j \in \{0, 1, \dots, b-1\}$ for all $j \in \mathbf{N}_0$. The k -th b -adic Walsh function ${}_b\text{wal}_k : [0, 1) \rightarrow \{0, \omega_b, \dots, \omega_b^{b-1}\}$ is defined as

$${}_b\text{wal}_k(x) := \omega_b^{\kappa_0 x_1 + \kappa_1 x_2 + \cdots},$$

for $x \in [0, 1)$ with b -adic expansion $x = x_1 b^{-1} + x_2 b^{-2} + x_3 b^{-3} + \cdots$ with $x_j \in \{0, 1, \dots, b-1\}$, which is unique in the sense that infinitely many of the x_j must be different from $b-1$.

This definition is generalized to the higher-dimensional case.

DEFINITION 5. For dimension $s \geq 1$, let $\mathbf{k} = (k_1, \dots, k_s) \in \mathbf{N}_0^s$. The $\mathbf{k}$ -th b -adic Walsh function ${}_b\text{wal}_{\mathbf{k}} : [0, 1)^s \rightarrow \{0, \omega_b, \dots, \omega_b^{b-1}\}$ is defined as

$${}_b\text{wal}_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^s {}_b\text{wal}_{k_i}(x_i).$$

for $\mathbf{x} = (x_1, \dots, x_s) \in [0, 1)^s$.

Walsh coefficients are defined as follows.

DEFINITION 6. Let $f : [0, 1)^s \rightarrow \mathbf{R}$. The $\mathbf{k}$ -th b -adic Walsh coefficient of f is defined as

$$\mathcal{F}(f)(\mathbf{k}) := \int_{[0, 1)^s} f(\mathbf{x}) \overline{{}_b\text{wal}_{\mathbf{k}}(\mathbf{x})} d\mathbf{x}.$$

We see the relationship between Walsh coefficients and discrete Fourier coefficients in the following. Let $A = (a_{i,j}) \in \mathbf{Z}_b^{s \times n}$. We define maps $\phi_i : \mathbf{Z}_b^{s \times n} \rightarrow \mathbf{N}_0$ as $\phi_i(A) = \sum_{j=1}^n a_{i,j} b^{j-1}$ and $\phi : \mathbf{Z}_b^{s \times n} \rightarrow \mathbf{N}_0^{ts}$ as $\phi(A) = (\phi_1(A), \dots, \phi_s(A))$. Note that $\phi_i(A) < b^n$ holds for all $1 \leq i \leq s$ and $A \in \mathbf{Z}_b^{s \times n}$.

LEMMA 2. Let $f : [0, 1)^s \rightarrow \mathbf{R}$ and $A = (a_{i,j}) \in \mathbf{Z}_b^{s \times n}$. Then we have

$$\overline{\mathcal{F}(f)(\phi(A))} = \hat{f}_n(A).$$

PROOF. Since $\phi_i(A) < b^n$ holds for all $1 \leq i \leq s$, for all $\mathbf{x} = (x_1, \dots, x_s) \in \mathbf{I}_B$ we have

$${}_b\text{wal}_{\phi(A)}(\mathbf{x}) = \prod_{i=1}^s {}_b\text{wal}_{\phi_i(A)}(x_i) = \prod_{i=1}^s \omega_b^{a_{i,1}b_{i,1} + \dots + a_{i,n}b_{i,n}} = B \bullet A.$$

Therefore we have

$$\begin{aligned} \overline{\mathcal{F}(f)(\phi(A))} &= \int_{[0,1]^s} f(\mathbf{x}) {}_b\text{wal}_{\phi(A)}(\mathbf{x}) d\mathbf{x} \\ &= \sum_{B \in \mathbf{Z}_b^{s \times n}} \int_{\mathbf{I}_B} f(\mathbf{x}) {}_b\text{wal}_{\phi(A)}(\mathbf{x}) d\mathbf{x} \\ &= \sum_{B \in \mathbf{Z}_b^{s \times n}} \int_{\mathbf{I}_B} f(\mathbf{x})(B \bullet A) d\mathbf{x} \\ &= \sum_{B \in \mathbf{Z}_b^{s \times n}} (B \bullet A) \int_{\mathbf{I}_B} f(\mathbf{x}) d\mathbf{x} \\ &= \sum_{B \in \mathbf{Z}_b^{s \times n}} (B \bullet A) \cdot \text{Vol}(\mathbf{I}_B) f_n(B) \\ &= \sum_{B \in \mathbf{Z}_b^{s \times n}} (B \bullet A) \cdot b^{-sn} f_n(B) = \widehat{f}_n(A), \end{aligned}$$

which proves the lemma. $\square$

Let $f \sim \sum_{\mathbf{k} \in \mathbb{N}_0^s} \mathcal{F}(f)(\mathbf{k}) {}_b\text{wal}_{\mathbf{k}}$ be the Walsh expansion of a real valued function $f: [0,1]^s \rightarrow \mathbf{R}$. Lemma 2 implies that considering n -digit discrete approximation f_n of f is the same as considering the Walsh polynomial $\sum_{\mathbf{k} < b^n} \mathcal{F}(f)(\mathbf{k}) \cdot {}_b\text{wal}_{\mathbf{k}}$, where $\mathbf{k} = (k_1, \dots, k_s) < b^n$ means that $k_i < b^n$ holds for every $i = 1, \dots, s$, namely we have the following.

PROPOSITION 1. Let $f: [0,1]^s \rightarrow \mathbf{R}$. For $B \in \mathbf{Z}_b^{s \times n}$, we have $f_n(B) = \sum_{\mathbf{k} < b^n} \mathcal{F}(f)(\mathbf{k}) {}_b\text{wal}_{\mathbf{k}}(\mathbf{x}_B)$.

PROOF.

$$\begin{aligned} f_n(B) &= \sum_{A \in \mathbf{Z}_b^{s \times n}} \widehat{f}_n(A) B \bullet A \quad (\because \text{Theorem 2}) \\ &= \sum_{A \in \mathbf{Z}_b^{s \times n}} \mathcal{F}(f)(\phi(A)) {}_b\text{wal}_{\phi(A)}(\mathbf{x}_B) \quad (\because \text{Lemma 2}) \\ &= \sum_{\mathbf{k} < b^n} \mathcal{F}(f)(\mathbf{k}) {}_b\text{wal}_{\mathbf{k}}(\mathbf{x}_B). \end{aligned}$$

$\square$

3. WAFOM over a finite abelian group

In this section, we expand the notion of WAFOM defined in [10], more precisely, we define WAFOM over a finite abelian group with b elements.

First, we evaluate the n -th discretized QMC integration error of f with its discrete Fourier coefficients. Let $P \subset \mathbf{Z}_b^{s \times n}$ be a subgroup. We have $I(f) = \widehat{f}_n(O)$ by the definition of the discrete Fourier inversion, and we have $I_{P,n}(f) = \sum_{A \in P^\perp} \widehat{f}_n(A)$ by the Poisson summation formula (Theorem 1). Hence we have

$$\text{Err}(f; P, n) = |I_{P,n}(f) - I(f)| = \left| \sum_{A \in P^\perp \setminus \{O\}} \widehat{f}_n(A) \right| \leq \sum_{A \in P^\perp \setminus \{O\}} |\widehat{f}_n(A)|,$$

and thus we would like to bound the value $|\widehat{f}_n(A)|$. Dick gives an upper bound of the $\mathbf{k}$ -th b -adic Walsh coefficient $\mathcal{F}(f)(\mathbf{k})$ for n -smooth function f (for the definition of n -smoothness, see [3] or [6, §14]).

THEOREM 3 ([6, Theorem 14.23]). *There is a constant $C_{b,s,n}$ depending only on b, s and n such that for any n -smooth function $f : [0, 1]^s \rightarrow \mathbf{R}$ and any $\mathbf{k} \in \mathbf{N}^s$ it holds that*

$$|\mathcal{F}(f)(\mathbf{k})| \leq C_{b,s,n} \|f\|_n \cdot b^{-\mu_n(\mathbf{k})},$$

where $\|f\|_n$ is a norm of f for a Sobolev space and $\mu_n(\mathbf{k})$ is the n -weight of $\mathbf{k}$, which are defined in [6, (14.6) and Theorem 14.23] (we do not define them here).

Instead of μ_n , we define the Dick weight μ on dual groups of general finite abelian groups below, which is a generalization of the Dick weight over $\mathbf{F}_2$ defined in [10]. Actually, μ is a special case of $\mu_n \circ \phi$. More precisely, if $G = \mathbf{Z}_b$ and $\alpha \geq n$ hold, then we have $\mu = \mu_\alpha \circ \phi$ as a function from $(\mathbf{Z}_b^\vee)^{s \times n} (\simeq \mathbf{Z}_b^{s \times n})$ to $\mathbf{N}_0$.

DEFINITION 7. Let G be a finite abelian group and let $A \in (G^\vee)^{s \times n}$. The Dick weight $\mu : (G^\vee)^{s \times n} \rightarrow \mathbf{N}_0$ is defined as

$$\mu(A) := \sum_{i,j} j \times \delta(a_{i,j}),$$

with $\delta(h) = 0$ for $h = 0$ and $\delta(h) = 1$ for $h \neq 0$.

We obtain the next corollary.

COROLLARY 1. *There exists a constant $C_{b,s,n}$ depending only on b, s and n such that for any n -smooth function $f : [0, 1]^s \rightarrow \mathbf{R}$ and any $A \in (\mathbf{Z}_b^\vee)^{s \times n}$ it holds that*

$$|\widehat{f}_n(A)| \leq C_{b,s,n} \|f\|_n \cdot b^{-\mu(A)}.$$

PROOF. This is the direct corollary of Theorem 3, Lemma 2, and the equality $\mu(A) = \mu_n \circ \phi(A)$. $\square$

By the above corollary, we have a bound on the n -th discretized QMC integration error

$$\text{Err}(f; P, n) := |I(f) - I_{P,n}(f)| \leq C_{b,s,n} \|f\|_n \times \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)},$$

for a subgroup P of $\mathbf{Z}_b^{s \times n}$.

Hence, as a generalization of [10], we define a kind of figure of merit (the Walsh figure of merit or WAFOM).

DEFINITION 8. Let s, n be positive integers. Let G be a finite abelian group with b elements. Let $P \subset G^{s \times n}$ be a subgroup of $G^{s \times n}$. We define the Walsh figure of merit of P by

$$\text{WF}(P) := \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)}.$$

In order to stress the role of the precision n , we sometimes denote $\text{WF}^n(P)$ instead of $\text{WF}(P)$.

Then, as we have seen, we have the Koksma-Hlawka type inequality

$$\text{Err}(f; P, n) := |I(f) - I_{P,n}(f)| \leq C_{b,s,n} \|f\|_n \times \text{WF}(P)$$

for a subgroup $P \subset \mathbf{Z}_b^{s \times n}$. This shows that $\text{WF}(P)$ is a quality measure of the point set P for quasi-Monte Carlo integration when $G = \mathbf{Z}_b$.

4. MacWilliams identity over an abelian group

In this section, we assume that s, n are positive integers. Recall that G is a finite abelian group and $G^\vee$ its character group. We consider an abelian group $G^{s \times n}$. Let $P \subset G^{s \times n}$ be a subgroup.

We are interested in the weight enumerator polynomial of $P^\perp$

$$W_{P^\perp}(x, y) := \sum_{A \in P^\perp} x^{M - \mu(A)} y^{\mu(A)} \in \mathbf{C}[x, y],$$

where $M := n(n+1)s/2$.

Let $R := \mathbf{C}[x_{i,j}(h)]$, where $x_{i,j}(h)$ is a family of indeterminates for $1 \leq i \leq s$, $1 \leq j \leq n$, and $h \in G^\vee$. We define functions $f_{i,j} : G^\vee \rightarrow R$ as $f_{i,j}(h) = x_{i,j}(h)$ and $f : (G^{s \times n})^\vee = (G^\vee)^{s \times n} \rightarrow R$ as

$$f(A) := \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} f_{i,j}(a_{i,j}) = \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} x_{i,j}(a_{i,j}).$$

Now the complete weight enumerator polynomial of $P^\perp$, in a standard sense [8, Chapter 5], is defined by

$$GW_{P^\perp}(x_{i,j}(h)) := \sum_{A \in P^\perp} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} x_{i,j}(a_{i,j}),$$

and similarly, the complete weight enumerator polynomial of P is defined by

$$GW_P^*(x_{*i,j}(g)) := \sum_{B \in P} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} x_{*i,j}(b_{i,j})$$

in $R^* := \mathbf{C}[x_{*i,j}(g)]$ where $x_{*i,j}(g)$ is a family of indeterminates for $1 \leq i \leq s$, $1 \leq j \leq n$, and $g \in G$. We note that if we substitute

$$x_{i,j}(0) \leftarrow x^j, \quad x_{i,j}(h) \leftarrow y^j \quad \text{for } h \neq 0, \quad (1)$$

we have an identity

$$GW_{P^\perp}(x_{i,j}(h))|_{\text{above substitution}} = W_{P^\perp}(x, y).$$

A standard formula of the Fourier transform tells that, if $f_1 : G_1 \rightarrow R$, $f_2 : G_2 \rightarrow R$ are functions and $f_1 f_2 : G_1 \times G_2 \rightarrow R$ is their multiplication at the value, then

$$\widehat{f_1 f_2} = \widehat{f_1} \widehat{f_2}$$

holds. This implies that

$$\widehat{f}(B) = \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} \widehat{f_{i,j}}(b_{i,j}) = \frac{1}{|G|^{sn}} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} \sum_{h \in G^\vee} f_{i,j}(h)(h \bullet b_{i,j}).$$

Hence, by the Poisson summation formula (Theorem 1), we have

$$\begin{aligned} GW_{P^\perp}(x_{i,j}(h)) &= \sum_{A \in P^\perp} f(A) \\ &= |P^\perp| \sum_{B \in P} \widehat{f}(B) \\ &= \frac{1}{|P|} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} \sum_{h \in G^\vee} f_{i,j}(h)(h \bullet b_{i,j}). \end{aligned}$$

Thus we have the MacWilliams identity below, which is a variant of Generalized MacWilliams identity [8, Chapter 5 §6]:

PROPOSITION 2 (MacWilliams identity).

$$GW_{P^\perp}(x_{i,j}(h)) = \frac{1}{|P|} GW_P^* (\text{substituted}),$$

where in the right hand side every $x_{*i,j}(g)$ is substituted by

$$x_{*i,j}(g) \leftarrow \sum_{h \in G^\vee} (h \bullet g) x_{i,j}(h).$$

We consider specializations of this identity. First, we consider a specialization $\overline{GW}_{P^\perp}(x_1, \dots, x_n, y_1, \dots, y_n)$ of $GW_{P^\perp}(x_{i,j}(h))$ obtained by the substitution

$$x_{i,j}(0) \leftarrow x_j, \quad x_{i,j}(h) \leftarrow y_j \quad \text{for } h \neq 0.$$

We have

$$\begin{aligned} \sum_{h \in G^\vee} (h \bullet g) x_{i,j}(h) \Big|_{\text{above substitution}} &= (0 \bullet g) x_j + \sum_{h \in G^\vee \setminus \{0\}} (h \bullet g) y_j \\ &= x_j - y_j + \sum_{h \in G^\vee} (h \bullet g) y_j \\ &= x_j - y_j + \begin{cases} b y_j & (\text{if } g = 0) \\ 0 & (\text{otherwise}) \end{cases} \\ &= \begin{cases} x_j + (b-1) y_j & (\text{if } g = 0) \\ x_j - y_j & (\text{otherwise}) \end{cases}, \end{aligned}$$

where we use Lemma 1 for the third equality. Thus, we have the following formula.

COROLLARY 2.

$$\overline{GW}_{P^\perp}(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} (x_j + \eta(b_{i,j}) y_j),$$

where $\eta(b_{i,j}) = b-1$ if $b_{i,j} = 0$ and $\eta(b_{i,j}) = -1$ if $b_{i,j} \neq 0$.

Second, we consider the specialization (1) of $GW_{P^\perp}$. We have already seen that $GW_{P^\perp}|_{(\text{substitution (1)})} = W_{P^\perp}(x, y)$ holds. Since

$$W_{P^\perp}(x, y) = \overline{GW}_{P^\perp}(x^1, \dots, x^n, y^1, \dots, y^n)$$

follows, Corollary 2 implies the following formula:

THEOREM 4.

$$W_{P^\perp}(x, y) = \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} (x^j + \eta(b_{i,j})y^j),$$

where $\eta(b_{i,j}) = b - 1$ if $b_{i,j} = 0$ and $\eta(b_{i,j}) = -1$ if $b_{i,j} \neq 0$.

Using Theorem 4, we can compute $\text{WF}(P)$ and $\delta_{P^\perp}$, the minimum Dick weight of $P^\perp$. The minimum Dick weight of $P^\perp$ is defined as

$$\delta_{P^\perp} := \min_{B \in P^\perp \setminus \{O\}} \mu(B),$$

which is used for bounding WAFOM (see Section 5.3). First, we introduce how to compute $\text{WF}(P)$. The following formula to compute WAFOM is a generalization of [10, Corollary 4.2], which treats the case $G = \mathbf{F}_2$.

COROLLARY 3. *Let $P \subset \mathbf{Z}_b^{s \times n}$ be a subgroup. Then we have*

$$\text{WF}(P) = -1 + \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} (1 + \eta(b_{i,j})b^{-j}).$$

PROOF.

$$\begin{aligned} \text{WF}(P) &= \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \\ &= -1 + \sum_{A \in P^\perp} b^{-\mu(A)} \\ &= -1 + W_{P^\perp}(1, b^{-1}) \\ &= -1 + \frac{1}{|P|} \sum_{B \in P} \prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n}} (1 + \eta(b_{i,j})b^{-j}). \quad \square \end{aligned}$$

The merit of Theorem 4 and Corollary 3 is that the number of summation depends only on $|P|$ linearly, not $|P^\perp| = b^{sn}/|P|$. We can calculate weight enumerator polynomials by sn times multiplication between an integer polynomial with a binomial, and $|P|$ times addition of such polynomials of degree $n(n+1)/2$. In the case of computing WAFOM, we need sn times of multiplication of real numbers and $|P|$ times of summation of such real numbers, thus we need $O(sn|P|)$ times of operations of real numbers. On the other hand, to calculate weight enumerator polynomials based on the definition, we need $|P^\perp|$ times of summations of monomials, and to calculate weight WAFOM based on the definition, we need $|P^\perp|$ times of summations of real numbers.

For QMC, the size $|P|$ cannot exceed a reasonable number of computer operations, so $|P^\perp| = b^{sn}/|P|$ can be large if sn is sufficiently large. This implies that the computational complexity of calculating weight enumerator polynomials or WAFOM using Theorem 4 or Corollary 3 is smaller if sn is large.

Second, we introduce how to compute $\delta_{P^\perp}$. The minimum Dick weight $\delta_{P^\perp}$ is equal to the degree of leading nonzero term of $-1 + W_{P^\perp}(1, y)$, namely:

LEMMA 3. *Let $W_{P^\perp}(1, y) = 1 + \sum_{i=1}^{\infty} a_i y^i$. Then we have $\delta_{P^\perp} = \min\{i \mid a_i \neq 0\}$.*

Thus we can obtain the minimum Dick weight of $P^\perp$ by calculating the weight enumerator polynomial of $P^\perp$.

REMARK 1. *Because of Lemma 8 in Section 5.5, in order to compute $\delta_{P^\perp}$ it is sufficient to compute $W_{P^\perp}(1, y)$ only up to degree $\delta_{P^\perp} \leq d^2/(2s) + 3d/2 + s$.*

5. Estimation of WAFOM

The following arguments from Section 5.1 to Section 5.4 are generalizations of [11] which deals with the case $G = \mathbf{F}_2$, and arguments in Section 5.5 are generalizations of [19], which deals with the case $G = \mathbf{F}_2$. The methods for proofs are similar to [11] and [19]. In this section, we suppose that s and n are positive integers and that G is a finite abelian group.

5.1. Geometry of the Dick weight. Recall that G is a finite abelian group with $b \geq 2$ elements, $G^\vee$ its character group. The Dick weight $\mu : (G^\vee)^{s \times n} \rightarrow \mathbf{N}_0$ induces a metric

$$d(A, B) := \mu(A - B) \quad \text{for } A, B \in (G^\vee)^{s \times n}$$

and thus $(G^\vee)^{s \times n}$ can be regarded as a metric space.

Let $S_{s,n}(m) := |\{A \in (G^\vee)^{s \times n} \mid \mu(A) = m\}|$, namely $S_{s,n}(m)$ is the cardinality of the sphere in $(G^\vee)^{s \times n}$ with center 0 and radius m . A combinatorial interpretation of $S_{s,n}(m)$ is as follows. One has $s \times n$ dice. Each die has b faces. For each value $i = 1, \dots, n$, there exist exactly s dice with value 0 on one face and i on the other $b - 1$ faces. Then, $S_{s,n}(m)$ is the number of ways that the summation of the upper surfaces of $s \times n$ dice is m . This combinatorial interpretation implies the following identity:

$$\prod_{k=1}^n (1 + (b-1)x^k)^s = \sum_{m=0}^{\infty} S_{s,n}(m)x^m.$$

You can also see this identity from Theorem 4 for $P = \{O\}$, $x \leftarrow 1$, and $y \leftarrow x$. Note that the right hand side is a finite sum. It is easy to see that $S_{s,n}(m)$ is monotonically increasing with respect to s and n , and $S_{s,m}(m) = S_{s,m+1}(m) = S_{s,m+2}(m) = \cdots$ holds.

DEFINITION 9. $S_s(m) := S_{s,m}(m)$.

We have the following identity between formal power series:

$$\prod_{k=1}^{\infty} (1 + (b-1)x^k)^s = \sum_{m=0}^{\infty} S_s(m)x^m. \quad (2)$$

For any positive integer M , we define

$$\mathcal{B}_{s,n}(M) := \{A \in (G^\vee)^{s \times n} \mid \mu(A) \leq M\}, \quad \text{vol}_{s,n}(M) := |\mathcal{B}_{s,n}(M)|,$$

namely $\mathcal{B}_{s,n}(M)$ is the ball in $G^{s \times n}$ with center 0 and radius M , and $\text{vol}_{s,n}(M)$ is its cardinality. We have $\text{vol}_{s,n}(M) = \sum_{m=0}^M S_{s,n}(m)$, and thus $\text{vol}_{s,n}(M)$ inherits properties of $S_{s,n}(m)$, namely, $\text{vol}_{s,n}(M)$ is also monotonically increasing with respect to s and n , and $\text{vol}_{s,M}(M) = \text{vol}_{s,M+1}(M) = \text{vol}_{s,M+2}(M) = \cdots$ holds.

DEFINITION 10. $\text{vol}_s(M) := \text{vol}_{s,M}(M)$.

5.2. Combinatorial inequalities.

LEMMA 4.

$$\text{vol}_{s,n}(M) \leq \text{vol}_s(M) \leq \exp(2\sqrt{(b-1)sM}).$$

PROOF. We have already seen the first inequality. We prove the next inequality along [9, Exercise 3(b), p. 332], which treats only $S = 1$ and $b = 2$ case. If $M = 0$ it is trivial, and so we assume that $M > 0$. Define a polynomial with non-negative integer coefficients by

$$f_{s,M}(x) := \prod_{k=1}^M (1 + (b-1)x^k)^s.$$

Since $f_{s,M}(x)$ has only non-negative coefficients, from Identity (2) we have $\sum_{m=0}^M S_s(m)x^m \leq f_{s,M}(x)$ ($x \in (0, 1)$). Hence we have

$$\text{vol}_s(M) = \sum_{m=0}^M S_s(m) \leq \sum_{m=0}^M S_s(M)x^{m-M} \leq f_{s,M}(x)/x^M \quad (x \in (0, 1)).$$

By taking the logarithm of the both sides and using the well-known inequality $\log(1 + X) \leq X$, for all $x \in (0, 1)$ we have

$$\begin{aligned} \text{vol}_{s,n}(M) &\leq s \sum_{k=1}^M \log(1 + (b-1)x^k) + M \log(1/x) \\ &< s(b-1) \sum_{k=1}^M x^k + M \log\left(1 + \frac{1-x}{x}\right) \\ &< s(b-1) \frac{x}{1-x} + M \frac{1-x}{x}. \end{aligned}$$

By comparison of the arithmetic mean and the geometric mean, the last expression attains the minimum value $2\sqrt{(b-1)sM}$ when $s(b-1)x/(1-x) = M(1-x)/x$ holds, namely $x = (1 + \sqrt{(b-1)s/M})^{-1} \in (0, 1)$. $\square$

LEMMA 5.

$$S_{s,n}(M) \leq S_s(M) \leq \exp(2\sqrt{(b-1)sM}).$$

PROOF. It follows from Lemma 4 and the inequality $S_s(M) \leq \text{vol}_s(M)$. $\square$

5.3. Bounding WAFOM by the minimum weight.

DEFINITION 11. Let $P \subset G^{s \times n}$ be a subgroup. The minimum Dick weight of $P^\perp$ is defined by

$$\delta_{P^\perp} := \min_{B \in P^\perp \setminus \{O\}} \mu(B)$$

The next lemma bounds $\text{WF}(P)$ by the minimum weight of $P^\perp$.

LEMMA 6. For a positive integer M , define

$$C_{s,n}(M) := \sum_{A \in (G^\vee)^{s \times n} \setminus \mathcal{B}_{s,n}(M-1)} b^{-\mu(A)} = \sum_{m=M}^{\infty} S_{s,n}(m) b^{-m}$$

and

$$C_s(M) := \sum_{m=M}^{\infty} S_s(m) b^{-m}.$$

Then we have

$$\text{WF}^n(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \leq C_{s,n}(\delta_{P^\perp}) \leq C_s(\delta_{P^\perp}).$$

PROOF. The last inequality is trivial, so it suffices to prove the first inequality. Since $P^\perp \setminus \{O\} \subset (G^\vee)^{s \times n} \setminus \mathcal{B}_{s,n}(\delta_{P^\perp} - 1)$ holds, we have

$$\begin{aligned} \text{WF}^n(P) &= \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \leq \sum_{A \in (G^\vee)^{s \times n} \setminus \mathcal{B}_{s,n}(\delta_{P^\perp} - 1)} b^{-\mu(A)} \\ &= C_{s,n}(\delta_{P^\perp}). \end{aligned} \quad \square$$

We shall estimate $C_s(\lceil M' \rceil)$ (C for the Complement of the ball) for rather general real number M' : from Lemma 5 it follows that

$$\begin{aligned} C_s(\lceil M' \rceil) &= \sum_{m=\lceil M' \rceil}^{\infty} S_s(m) b^{-m} \\ &\leq \sum_{m=\lceil M' \rceil}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\ &= b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} + \sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}}. \end{aligned} \quad (3)$$

First, we estimate the second term of the above. The function

$$\exp(2\sqrt{(b-1)sm}) b^{-m} = \exp(2\sqrt{(b-1)sm} - m \log b)$$

is monotonically decreasing with respect to m if

$$\begin{aligned} \frac{d}{dm} (2\sqrt{(b-1)sm} - m \log b) &\leq 0 \Leftrightarrow \frac{2(b-1)s}{2\sqrt{(b-1)sm}} - \log b \leq 0 \\ &\Leftrightarrow \sqrt{\frac{(b-1)s}{m}} \leq \log b \\ &\Leftrightarrow m \geq (\log b)^{-2}(b-1)s, \end{aligned}$$

hence we assume that $M' \geq (\log b)^{-2}(b-1)s$. Then, we have

$$\begin{aligned} &\sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\ &\leq \int_{m=\lceil M' \rceil}^{\infty} e^{-m \log b} e^{2\sqrt{(b-1)sm}} dm \\ &= \int_{m=\lceil M' \rceil}^{\infty} \exp \left(-(\log b) \left(\sqrt{m} - \frac{\sqrt{(b-1)s}}{\log b} \right)^2 + \frac{(b-1)s}{\log b} \right) dm \end{aligned}$$

$$\begin{aligned}
&\leq \int_{m=M'}^{\infty} \exp\left(-(\log b)\left(\sqrt{m} - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) dm \\
&= \int_{x=\sqrt{M'}}^{\infty} \exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) 2x dx.
\end{aligned}$$

In order to bound the last integral from above, for a positive number c we assume that $\sqrt{M'} \geq (1+c)\sqrt{(b-1)s}/\log b$ or equivalently $M' \geq (1+c)^2(\log b)^{-2}(b-1)s$. This assumption is stronger than the previous assumption $M' \geq (\log b)^{-2}(b-1)s$. Then, on the domain of integration $x \geq \sqrt{M'} \geq (1+c)\sqrt{(b-1)s}/\log b$, we have $cx \leq (1+c)(x - \sqrt{(b-1)s}/\log b)$. Hence the estimation continues:

$$\begin{aligned}
&\sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\
&\leq \int_{x=\sqrt{M'}}^{\infty} \exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) \\
&\quad \times 2\frac{1+c}{c}\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right) dx \\
&= \frac{1+c}{c} \frac{1}{\log b} \left[-\exp\left(-(\log b)\left(x - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) \right]_{x=\sqrt{M'}}^{\infty} \\
&= \frac{1+c}{c} \frac{1}{\log b} \exp\left(-(\log b)\left(\sqrt{M'} - \frac{\sqrt{(b-1)s}}{\log b}\right)^2 + \frac{(b-1)s}{\log b}\right) \\
&= \frac{1+c}{c} \frac{1}{\log b} \exp(-(\log b)M' + 2\sqrt{(b-1)sM'}) \\
&= \frac{1+c}{c} \frac{1}{\log b} b^{-M'} e^{2\sqrt{(b-1)sM'}}.
\end{aligned}$$

Second, we consider the first term of (3). We have already proved that $\exp(2\sqrt{(b-1)sm})b^{-m}$ is monotonically decreasing if $m \geq (\log b)^{-2}(b-1)s$, and thus the assumption $M' \geq (\log b)^{-2}(b-1)s$ implies

$$b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} \leq b^{-M'} e^{2\sqrt{(b-1)sM'}}.$$

Therefore we have

$$\begin{aligned}
C_s(\lceil M' \rceil) &\leq b^{-\lceil M' \rceil} e^{2\sqrt{(b-1)s\lceil M' \rceil}} + \sum_{m=\lceil M' \rceil+1}^{\infty} b^{-m} e^{2\sqrt{(b-1)sm}} \\
&\leq b^{-M'} e^{2\sqrt{(b-1)sM'}} + \frac{1+c}{c} \frac{1}{\log b} b^{-M'} e^{2\sqrt{(b-1)sM'}} \\
&= \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}}.
\end{aligned}$$

Now we proved:

PROPOSITION 3. *Let c be a positive real number. Let M' be a real number with $M' \geq (1+c)^2(\log b)^{-2}(b-1)s$. Then we have the following bound*

$$C_{s,n}(\lceil M' \rceil) \leq C_s(\lceil M' \rceil) \leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}}.$$

5.4. Existence of low-WAFOM point sets. We denote the probability of the event A by $\text{prob}[A]$. Let p_b be the smallest prime factor of b . Let d be a positive integer. Choose d matrices $B_1, \dots, B_d \in G^{s \times n}$ independently and uniformly at random. Let $P = \langle B_1, \dots, B_d \rangle \subset G^{s \times n}$ be the G -linear span of $B_1, \dots, B_d$, namely $P = \{g_1 B_1 + \dots + g_d B_d \mid g_1, \dots, g_d \in G\}$ where $g \in G$ acts on $B = (b_{ij})$ by $gB = (gb_{ij})$. Note that $|P| \leq b^d$.

REMARK 2. *If $G = \mathbf{Z}_b$, by the theory of invariant factor decomposition, we can say that there exist matrices $B'_1, \dots, B'_d$ such that $P' := \langle B'_1, \dots, B'_d \rangle$ includes P and becomes a free $\mathbf{Z}_b$ -module of rank d . Thus if $G = \mathbf{Z}_b$, we can replace “subgroup P ” in this subsection with a “digital net P ,” since in this subsection we consider only the existence of a subgroup which has large minimum Dick weight, and $P \subset P'$ implies that $\delta_{P^\perp} \leq \delta_{P'^\perp}$.*

First, we evaluate $\text{prob}[\text{perp}_L]$, where we define perp_L as the event that $B_1, \dots, B_d$ are all perpendicular to $L \in (G^\vee)^{s \times n}$.

LEMMA 7. *Let $L \in (G^\vee)^{s \times n}$ be a nonzero matrix. Then we have $\text{prob}[L \perp B] \leq 1/p_b$. Especially we have $\text{prob}[\text{perp}_L] \leq p_b^{-d}$.*

PROOF. We consider the map $(L \bullet) : G^{s \times n} \rightarrow \mathbf{C}, B \mapsto L \bullet B$. Then we have the surjective group homomorphism $G^{s \times n} \rightarrow \text{Im}(L \bullet)$, and thus $|\text{Im}(L \bullet)|$ divides $G^{s \times n}$. Moreover, since L is nonzero, $|\text{Im}(L \bullet)|$ is larger than 1. Hence we have $|\text{Im}(L \bullet)| \geq p_b$. Therefore we have $\text{prob}[L \perp B] = |\text{Im}(L \bullet)|^{-1} \leq 1/p_b$, and especially we have $\text{prob}[\text{perp}_L] = \text{prob}[L \perp B]^d \leq p_b^{-d}$. $\square$

Let M be a positive integer. We evaluate the probability of the event that $\delta_{P^\perp} \geq M$. We have

$$\begin{aligned}
\text{prob}[\delta_{P^\perp} \geq M] &= 1 - \text{prob}[\delta_{P^\perp} \leq M - 1] \\
&= 1 - \text{prob}[\exists L \in \mathcal{B}_{s,n}(M-1) \setminus \{O\} \text{ s.t. } L \in P^\perp] \\
&= 1 - \text{prob}[\exists L \in \mathcal{B}_{s,n}(M-1) \setminus \{O\} \text{ s.t. } L \perp B_1, \dots, L \perp B_d] \\
&= 1 - \text{prob}[\bigcup_{L \in \mathcal{B}_{s,n}(M-1) \setminus \{O\}} \text{perp}_L] \\
&\geq 1 - \sum_{L \in \mathcal{B}_{s,n}(M-1) \setminus \{O\}} \text{prob}[\text{perp}_L] \\
&\geq 1 - (\text{vol}_{s,n}(M-1) - 1) \cdot p_b^{-d} \\
&> 1 - \text{vol}_{s,n}(M-1) \cdot p_b^{-d}.
\end{aligned}$$

This shows:

PROPOSITION 4. *If $\text{vol}_{s,n}(M-1) \leq p_b^d$ holds, then there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ satisfying $\delta_{P^\perp} \geq M$.*

By Lemma 4, the condition of this proposition is satisfied if it holds that

$$e^{2\sqrt{(b-1)s(M-1)}} \leq p_b^d \Leftrightarrow M \leq \frac{(\log p_b)^2 d^2}{4(b-1)s} + 1. \quad (4)$$

Therefore we have the following sufficient condition on the existence of M .

PROPOSITION 5. *If $M \leq (\log p_b)^2 d^2 / (4(b-1)s) + 1$ holds, then Inequality (4) is satisfied, and hence there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ satisfying $\delta_{P^\perp} \geq M$.*

From now on, we define $\alpha_b := (\log p_b)/2$ and $M' := A^2 d^2 / ((b-1)s)$ where $A \leq \alpha_b$ and we take M to be $\lfloor M' + 1 \rfloor$ so that P with $|P| \leq b^d$ and $\delta_{P^\perp} \geq M$ exists. Then, by Proposition 3, we have the following upper bound of $\text{WF}(P)$:

PROPOSITION 6. *Let $\alpha_b := (\log p_b)/2$. Take a real number A with $A \leq \alpha_b$ and an arbitrary real number $c > 0$. Then for any positive integers s, n , and $d \geq (1+c)(b-1)s/(A \log b)$, there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ satisfying*

$$\text{WF}^n(P) \leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-A^2 d^2 / ((b-1)s)} e^{2Ad}.$$

PROOF. Define $M' := A^2 d^2 / ((b-1)s)$ and $M := \lfloor M' + 1 \rfloor$. By Proposition 5, there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ and $\delta_{P^\perp} \geq M$. For this P , from Lemma 6 and Proposition 3 we have

$$\begin{aligned}
\text{WF}(P) &\leq C_s(M) \\
&= C_s(\lceil M' \rceil) \\
&\leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-M'} e^{2\sqrt{(b-1)sM'}} \\
&= \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-A^2 d^2 / ((b-1)s)} e^{2Ad},
\end{aligned}$$

which proves the proposition. $\square$

In particular, take $A = \alpha_b$ and we have the next theorem.

THEOREM 5. *Let $\alpha_b := (\log p_b)/2$ and take an arbitrary real number $c > 0$. Then for any s , n , and $d \geq (1+c)(b-1)s/(\alpha_b \log b)$, there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ satisfying*

$$\text{WF}(P) \leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-\alpha_b^2 d^2 / ((b-1)s)} e^{2\alpha_b d}.$$

Applying Theorem 5 to the case $G = \mathbf{F}_2$, we can improve [11, Theorem 2 and Remark 5].

COROLLARY 4. *Let $\alpha := \alpha_2 = (\log 2)/2$ and take an arbitrary real number $c > 0$. Then for any n and $d \geq (1+c)s/(\alpha \log 2)$, there exists a linear subspace $P \subset \mathbf{F}_2^{s \times n}$ with $\dim P \leq d$ satisfying*

$$\text{WF}(P) \leq \left(1 + \frac{1+c}{c} \frac{1}{\log 2}\right) 2^{-\alpha^2 d^2 / s} e^{2\alpha d}.$$

REMARK 3. *Suzuki [18] proved that the construction of higher order digital nets on $\mathbf{F}_p$ given in [3] combined with some Niederreiter-Xing point sets [14] yields an explicit construction of low-WAFOM point sets, whose order of WAFOM is almost the same with the order obtained in this paper.*

5.5. A lower bound of WAFOM. In this subsection, we show a lower bound on $\text{WAFOM}(P)$, as a generalization of [19]. The next lemma gives an upper bound on the minimum Dick weight of $P^\perp$ for given $P \subset G^{s \times n}$, which implies a lower bound of $\text{WAFOM}(P)$.

LEMMA 8. *Suppose that s and n are positive integers. Let $P \subset G^{s \times n}$ be a subgroup with $|P| \leq b^d$. Let q, r be nonnegative integers which satisfy $d = qs + r$ and $0 \leq r < s$. Then we have the following:*

- (1) $\delta_{P^\perp} \leq sq(q+1)/2 + (q+1)(r+1) \leq d^2/2s + 3d/2 + s$.
- (2) *Let C be an arbitrary positive real number greater than $1/2$. If $d/s \geq (\sqrt{C+1/16} + 3/4)/(C-1/2)$ holds, then we have $\delta_{P^\perp} \leq Cd^2/s$.*

PROOF. We define a subgroup $Q := \{A = (a_{ij}) \in (G^\vee)^{s \times n} \mid a_{ij} = 0 \text{ if } (q+2 \leq j \leq n) \text{ or } (j = q+1 \text{ and } r+2 \leq i \leq s)\}$. We have $|Q| = b^{qs+r+1} = b^{d+1}$. There is a $\mathbf{Z}$ -module isomorphism $P^\perp / (P^\perp \cap Q) \simeq (P^\perp + Q)/Q$, and thus we have

$$|P^\perp \cap Q| = \frac{|P^\perp| \cdot |Q|}{|P^\perp + Q|} \geq \frac{b^{sn-d} \cdot b^{d+1}}{|(G^\vee)^{s \times n}|} = b,$$

especially there exists a non-zero matrix $A' \in (P^\perp \cap Q)$. Therefore we have

$$\delta_{P^\perp} \leq \mu(A') \leq \max\{\mu(A) \mid A = (a_{ij}) \in Q\} = sq(q+1)/2 + (q+1)(r+1),$$

where the last equality holds if the components of A is as follows:

$$\begin{cases} a_{ij} = 0 & \text{if } (q+2 \leq j \leq n) \text{ or } (j = q+1 \text{ and } r+2 \leq i \leq s) \\ a_{ij} \neq 0 & \text{if } (1 \leq j \leq q) \text{ or } (j = q+1 \text{ and } 1 \leq i \leq r+1) \end{cases}.$$

In particular, since $q \leq d/s$ and $r+1 \leq s$, we have

$$\begin{aligned} \delta_{P^\perp} &\leq sq(q+1)/2 + (q+1)(r+1) \\ &\leq \frac{d}{2} \left(\frac{d}{s} + 1 \right) + \left(\frac{d}{s} + 1 \right) s = \frac{d^2}{s} \left(\frac{1}{2} + \frac{3s}{2d} + \frac{s^2}{d^2} \right), \end{aligned}$$

which proves the first statement.

Let C be a real number greater than $1/2$ and we assume $d/s \geq (\sqrt{C+1/16} + 3/4)/(C-1/2)$. Then we have $1/2 + 3s/2d + s^2/d^2 \leq C$. Thus we obtain

$$\delta_{P^\perp} \leq \frac{d^2}{s} \left(\frac{1}{2} + \frac{3s}{2d} + \frac{s^2}{d^2} \right) \leq Cd^2/s,$$

which proves the second statement. □

The above lemma gives a lower bound of $\text{WF}(P)$.

THEOREM 6. *Suppose that s and n are positive integers. Let G be a finite abelian group with $b \geq 2$ elements. Let $P \subset G^{s \times n}$ be a subgroup with $|P| \leq b^d$. Let C be an arbitrary positive real number greater than $1/2$. If $d/s \geq (\sqrt{C+1/16} + 3/4)/(C-1/2)$ holds, then we have*

$$\text{WF}^n(P) \geq b^{-Cd^2/s}.$$

PROOF.

$$\text{WF}^n(P) = \sum_{A \in P^\perp \setminus \{O\}} b^{-\mu(A)} \geq b^{-\delta_{P^\perp}} \geq b^{-Cd^2/s}. \quad \square$$

5.6. Order of WAFOM. In this subsection, we consider the order of $\text{WF}(P)$ where P is a subgroup of $G^{s \times n}$ with $|P| = b^d$.

We fix the base b . Let $D := \alpha_b = (\log p_b)/2$. We fix a positive integer E satisfying $E > (b-1)/(D \log b)$. Let c be the real number such that $E = (1+c)(b-1)/(D \log b)$ (by the assumption that $E > (b-1)/(D \log b)$, c is positive). Note that c , D and E depend only on b .

We assume that $d/s \geq E$. Then, by Proposition 6, there exists a subgroup $P \subset G^{s \times n}$ with $|P| \leq b^d$ satisfying

$$\text{WF}^n(P) \leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) b^{-D^2 d^2 / ((b-1)s)} e^{2Dd}.$$

Moreover, by Theorem 6, for every P with $|P| \leq b^d$ we have $\text{WF}^n(P) \geq b^{-Cd^2/s}$ where $C = (1/2 + 3/(2E) + 1/E^2)$. Thus we have the following lemma.

LEMMA 9. *If $d/s \geq E$, we have*

$$\begin{aligned} -Cd^2/s &\leq \min\{\log_b(\text{WF}^n(P)) \mid P \subset G^{s \times n} \text{ subgroup, } |P| \leq b^d\} \\ &\leq -D^2 d^2 / ((b-1)s) + 2Dd/\log b + \log_b \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right). \end{aligned}$$

Especially, let $N = b^d$ and we have the following.

THEOREM 7. *Let G be a finite abelian group with $|G| = b$. Let $P \subset G^{s \times n}$ be a subgroup with $|P| \leq N$. Let c , C , D , and E are constants as Lemma 9, which depend only on b . Suppose that $(\log N)/s \geq E$. Then we have*

$$\begin{aligned} N^{-C(\log N)/s} &\leq \min\{\text{WF}^n(P) \mid P \subset G^{s \times n} \text{ subgroup, } |P| \leq N\} \\ &\leq \left(1 + \frac{1+c}{c} \frac{1}{\log b}\right) N^{-D^2(\log N)/((\log b)(b-1)s) + 2D/\log b}. \end{aligned}$$

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