Anisohedral spherical triangles and classification of spherical tilings by congruent kites, darts and rhombi

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Abstract. We classify all spherical monohedral (kite/dart/rhombus)-faced tilings, as follows: The set of spherical monohedral rhombus-faced tilings consists of (1) the central projection of the rhombic dodecahedron, (2) the central projection of the rhombic triacontahedron, (3) a series of non-isohedral tilings, and (4) a series of tilings which are topologically trapezohedra (here a trapezohedron is the dual of an antiprism). The set of spherical tilings by congruent kites consists of (1) the central projection of the tetragonal icosikaitetrahedron, (2) the central projection of the tetragonal hexacontahedron, (3) a non-isohedral tiling obtained from the tetragonal icosikaitetrahedron by gliding a hemisphere of $T$ with $\pi/4$ radian, and (4) a continuously deformable series of tilings which are topologically trapezohedra. The set of spherical tilings by congruent darts is a continuously deformable series of tilings which are topologically trapezohedra. In the above explanation, unless otherwise stated, the tilings we have enumerated are isohedral and admit no continuous deformation. We prove that if a spherical (kite/dart/rhombus) admits an edge-to-edge spherical monohedral tiling, then it also does a spherical isohedral tiling. We also prove that the set of anisohedral, spherical triangles (i.e., spherical triangles admitting spherical monohedral triangular tilings but not any spherical isohedral triangular tilings) consists of a certain, infinite series of isosceles triangles $I_n$ and an infinite series of right scalene triangles which are the bisections of $I_n$.

1. Introduction

Ueno-Agaoka [16] classified all spherical tilings by congruent triangles. Here, by a tiling, we mean an edge-to-edge tiling without two-valent vertices. This paper presents two applications of their classification.

As the first application, we prove that the set of anisohedral, spherical triangles consists of an infinite series of spherical isosceles triangles $AI_n$ ($5 \neq n \geq 4$) and an infinite series of spherical right scalene triangles $ARS_m$ ($m \geq 4$), each of which is the bisection of $AI_n$. See Figure 7 and Theorem 1. Here a tile $T$ of a monohedral tiling is said to be anisohedral, provided that

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for every monohedral tiling $\mathcal{T}$ with the tile being congruent to $T$, $\mathcal{T}$ is not isohedral. A tiling on a constant curvature space is said to be monohedral, if the tiling consists of congruent tiles, while a tiling is said to be isohedral (or tile-transitive), if the symmetry group of the tiling acts transitively on the set of tiles. Monohedral tilings of the Euclidean space $\mathbb{R}^d$ have long been studied (see [10] for the survey). In his famous eighteenth problem, Hilbert posed three problems. The second problem of the three is

\((\dagger)\) “are there any anisohedral tile of the Euclidean space $\mathbb{R}^3$?” [12, 3].

As for the problem with the Euclidean space replaced by the two-dimensional sphere, the answer is yes. That is, there are anisohedral, spherical triangles, and they are the following: Recall that any spherical triangle is determined uniquely modulo congruence by the three inner angles [2, p. 62].

**Theorem 1.** The set of anisohedral, spherical triangles consists of

1. an infinite series of isosceles triangles $A_{I_n}$ such that the list of three inner angles is
   
   \[
   \left(\frac{\pi}{2} - \frac{1}{2n}, \frac{\pi}{2} - \frac{1}{2n}, \frac{2\pi}{n}\right) \quad (5 \neq n \geq 4);
   \]

2. an infinite series of right scalene triangles $A_{RS_m}$, which is the bisection of $A_{I_n}$. The list of three inner angles of $A_{RS_m}$ is
   
   \[
   \left(\frac{\pi}{2}, \frac{1}{2} - \frac{1}{2m}, \frac{\pi}{m}\right) \quad (m \geq 4).
   \]

The second application of Ueno-Agaoka’s complete classification [16] of spherical monohedral triangular tilings is a complete classification (Theorem 2) of all spherical tilings by congruent kites, all spherical tilings by congruent darts, and all spherical tilings by congruent rhombi. Here, by a kite (dart, resp.), we mean a convex (non-convex, resp.) quadrangle such that the cyclic list of edge-lengths is $aabb$ ($a \neq b$). Figure 1 is the list of all spherical monohedral (kite/dart/rhombus)-faced tilings. Each tiling in a square frame is an instance of a continuously deformable tiling, while the other tilings admit no continuous deformation. The tilings left to the double strokes are sporadic, and the others are instances of series. All the tilings in Figure 1 are isohedral except a spherical monohedral kite-faced tiling $gK_{24}$ (see the image in the topmost leftmost oval frame in Figure 1) and a spherical monohedral rhombus-faced tiling $gTR_{4n+2}$ ($n \geq 2$; see the image in the bottom rightmost oval frame in Figure 1). However, the tile of the spherical monohedral, non-isohedral kite-faced tiling $gK_{24}$ is congruent to the tile of a spherical isohedral kite-faced tiling $K_{24}$ (see the image next to the image of $gK_{24}$ in Figure 1), and the tile of
the other spherical monohedral, non-isohedral rhombus-faced tiling \( gTR_{4n+2} \) is congruent to the tile of a spherical isohedral rhombus-faced tiling \( TR_{2n/(2n+1)} \) (see the image next to the image of \( gTR_{4n+2} \) in Figure 1). The definitions and the symbols for the tilings are given in the next section.

The consequences of the classification of spherical monohedral (kite/dart/rhombus)-faced tilings are: (1) none of spherical kites, spherical darts, and spherical rhombi is anisohedral. This is a slight refinement of the affirmative solution (Theorem 1) of Hilbert’s question (1) on an anisohedral tile in his eighteenth problem adapted for the two-dimensional sphere. (2) In order to classify all spherical monohedral quadrangular tilings, it is enough to classify those consisting of congruent quadrangles such that the cyclic list of edge-lengths of the tile is \( aaab \) or \( aabc \) (\( a, b, c \) are mutually unequal), because of Ueno-Agaoka [15]:

**Proposition 1.** In any spherical tilings consisting of congruent quadrangles, the tile necessarily has at least two equilateral adjacent edges.

For an attempt to classify such spherical tilings, see [1].

The organization of this paper is as follows: In the next section, for each spherical monohedral (kite/dart/rhombus)-faced tilings appearing in Table 1, we give an explicit construction of it. Some constructions correspond to be “kis”ing [4, Chapter 21] for spherical tilings. This helps to show the isohedrality of spherical monohedral triangular tilings in Section 4. In Section 3, we state our main classification result (Theorem 2 including Table 1 and Figure 1). In Subsection 3.1 (Subsection 3.2, resp.), we prove that (1) Table 1 lists all
spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings, and (2) for each spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling \( \mathcal{Q} \), we provide values to each inner angle of the graph, represent the symmetry of \( \mathcal{Q} \) by a Schönflies symbol [5, p. 41], and decide whether the spherical (kite/dart)-faced (rhombus-faced, resp.) tiling \( \mathcal{Q} \) is isohedral or not. In Subsection 3.2, we also determine the spherical coordinates of the vertices and the length of the edges of the spherical non-isohedral monohedral rhombus-faced tiling. In Section 4, we identify all the anisohedral, spherical triangles by using Proposition 2, and prove that there are no anisohedral, spherical (kites/darts/rhombi).

By the sphere, we mean the unit sphere \( \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \} \). Unless otherwise stated, the unit for inner angles is \( \pi \) radian, while that for edge-length is radian. Ueno-Agaoka’s classification (Table of [16]) of all spherical monohedral triangular tilings is included and is referred to as Table 2. The subscript of the name of a tiling indicates the number of the tiles, and the names of spherical monohedral triangular tilings are from Table 2. A spherical monohedral triangular tiling \( H_{20} \) in Table 2 is the central projection of the regular icosahedron to the sphere [16, p. 485]. Let \( C_8 \) (\( D_{12} \), resp.) be the central projection of the regular octahedron (the regular dodecahedron, resp.) to the sphere.

2. Explicit constructions of the spherical monohedral (kite/dart/rhombus)-faced tilings

We will explicitly construct

- three spherical monohedral kite-faced tilings \( K_{60}, K_{24}, gK_{24} \),
- a deformable spherical monohedral (kite/dart/rhombus)-faced tiling \( TR_{2n}^\alpha \) (\( n \geq 3 \)) \( (1/(2n) < \alpha < 1 - 1/(2n), \alpha \neq 1/2) \), and
- three spherical monohedral rhombus-faced tilings \( R_{12}, R_{30}, gTR_{4n+2} \) (\( n \geq 2 \)).

We will construct the two spherical monohedral kite-faced tilings \( K_{60} \) and \( K_{24} \), from two spherical tilings \( F_{120} \) and \( F_{48} \) by congruent right scalene triangles, and will construct the two spherical monohedral rhombus-faced tilings \( R_{12} \) and \( R_{30} \), from two spherical tilings \( F_{24} \) and \( F_{60}^I \) by congruent isosceles triangles. Here \( F_{120}, F_{48}, F_{24} \) and \( F_{60}^I \) are listed in Ueno-Agaoka’s complete classification ([16, Table], included as Table 2). According to [16, Figure 17 and p. 484], \( F_{24} \) is \( \text{kis}(C_8) \) and \( F_{60}^I \) is \( \text{kis}(H_{20}) \) where \( \text{kis} \) is the following construction.

**Definition 1.** Let \( \mathcal{T} \) be a spherical tiling. The “\( \text{kis} \)”ing of \( \mathcal{T} \) is a spherical tiling by triangles obtained from \( \mathcal{T} \) by joining the center of each
tile to each vertex of that tile. The resulting tiling is denoted by \( \text{kis}(\mathcal{T}) \). If \( \mathcal{T} \) is a spherical tiling by \( n \) congruent rhombi, then \( \text{kis}(\mathcal{T}) \) is a spherical tiling by \( 4n \) congruent right scalene triangles. If \( \mathcal{T} \) is a spherical tiling by \( n \) regular \( p \)-gons, then \( \text{kis}(\mathcal{T}) \) is a spherical tiling by \( pn \) congruent isosceles triangles.

The “kis”ing construction of spherical monohedral triangular tilings corresponds to the “kis”ing construction [4, Chapter 21] of Archimedean duals. In [4, Chapter 21], each Archimedean dual obtained via the “kis”ing of a Platonic solid or another Archimedean dual is named “kis...”

According to Ueno-Agaoka [16, Figure 21], a spherical tiling \( F_{120} (F_{48}, \text{ resp.}) \) by congruent right scalene triangles, listed in Table 2, is constructed from \( \mathcal{D}_{12} (\mathcal{C}_8, \text{ resp.}) \), by \( (H) \) joining each center \( c \) of each face \( \Phi \) of \( \mathcal{D}_{12} (\mathcal{C}_8, \text{ resp.}) \) to each vertex \( v \) of the face \( \Phi \), and \( (C) \) joining \( c \) to the midpoint \( m \) of each edge of \( \Phi \). Each edge \( (H) \) is the hypotenuse of the right scalene triangle of \( F_{120} (F_{48}, \text{ resp.}) \). \( F_{120} (F_{48}, \text{ resp.}) \) is a barycentric subdivision [8] (or also the chamber system) of \( \mathcal{D}_{12} (\mathcal{C}_8, \text{ resp.}) \).

2.1. \( K_{60} \) and \( K_{24} \). A spherical monohedral kite-faced tiling \( K_{60} (K_{24}, \text{ resp.}) \) is constructed from the spherical monohedral triangular tiling \( F_{120} (F_{48}, \text{ resp.}) \), by deleting the hypotenuse (cf. \( (H) \)) of each right scalene triangular tile. In other words, \( K_{60} (K_{24}, \text{ resp.}) \) is constructed from \( \mathcal{D}_{12} (\mathcal{C}_8, \text{ resp.}) \) by joining each center to the midpoints of the surrounding edges.

\( K_{24} \) is indeed a spherical monohedral kite-faced tiling, because of the following argument: All hypotenuses of the tiles of \( F_{48} \) are of equal length and bisect the inner angles \( 2/3 \) between two adjacent edges obtained by \( (C) \), as well as an inner angle \( 1/2 \) of \( \mathcal{C}_8 \). Hence, deleting the hypotenuses of the tiles of \( F_{48} \) results in a spherical monohedral tiling. The tile of \( K_{24} \) is a kite, because the tile has a pair \( (2/3, 1/2) \) as opposite inner angles on the two vertices of the edge \( (H) \). However, the other pair of opposite inner angles is \( (1/2, 1/2) \).

\( K_{60} \) is a spherical monohedral kite-faced tiling, by a similar argument.

We will prove that the symmetry of \( K_{24} \) is that of the regular octahedron, in order to assure that a construction (Subsection 2.2) of a spherical monohedral kite-faced tiling \( gK_{24} \) is well-defined. The symmetry of the regular octahedron (the regular icosahedron, resp.) is represented by a Schönflies symbol \( O_h \) \( (I_h, \text{ resp.}) \) [5, p. 48 (p. 49, resp.)].

**Lemma 1.** The Schönflies symbol of \( K_{24} \) \( (K_{60}, \text{ resp.}) \) is \( O_h \) \( (I_h, \text{ resp.}) \).

**Proof.** Every symmetry operation \( \sigma \) of \( \mathcal{C}_8 \) \( (\mathcal{D}_{12}, \text{ resp.}) \) preserves the triangular (pentagonal, resp.) tiles, the centers of the triangular (pentagonal,
resp.) tiles and the midpoints of the edges of $\mathcal{O}_8$ ($D_{12}$, resp.). Hence the Schönflies symbol of $K_{24}$ ($K_{60}$, resp.) is $O_h$ ($I_h$, resp.), by [5, p. 50].

2.2. $gK_{24}$. A spherical monohedral tiling $gK_{24}$ is constructed as follows: $K_{24}$ has three mutually perpendicular, great circles. Each great circle consists of eight edges of $K_{24}$ and is the intersection of $K_{24}$ and some mirror plane of $K_{24}$. Choose one of the three great circles, and then a hemisphere $\mathcal{N}$ of $K_{24}$ determined by the chosen great circle. Any choice of $\mathcal{N}$ results in a congruent spherical figure. It is because any great circle of $K_{24}$ is transformed to any other great circle of $K_{24}$ by some symmetry operation of $K_{24}$, by Lemma 1. Glide the hemisphere $\mathcal{N}$ by $\pi/4$ radian against the other hemisphere. Let the resulting spherical figure be $gK_{24}$. Because the boundary of the hemisphere $\mathcal{N}$ is equally subdivided into eight edges of $K_{24}$, the $\pi/4$ radian rotation keeps the tiling being edge-to-edge. See Figure 2. $gK_{24}$ is a spherical monohedral kite-faced tiling, since the tiles are not changed by this gliding.

2.3. $TR_{2n}^n$. A spherical monohedral tiling $TR_{2n}^n$ ($n \geq 3$) is constructed as in Figure 3 so that each pole is shared by $n$ tiles. $TR_{2n}^n$ is topologically the so-called $n$-gonal trapezohedron of $2n$ faces [11]. Here an $n$-gonal trapezohedron is the dual of an $n$-gonal antiprism. Let the cyclic list of four inner angles of the tile of $TR_{2n}^n$ be $(\alpha, \beta, \gamma, \delta)$ such that $\beta = 2/n$ and $\delta = 2 - 2\alpha$, as in Figure 3 (left, right). Note $\alpha \neq 1/2$. Otherwise, $\delta = 1$ and thus the tile of $TR_{2n}^n$ is a triangle.

Fig. 2. The left is a graph of the spherical monohedral kite-faced tiling $gK_{24}$ by 24 congruent kites (thick and dotted). The set of the thick, dotted, and thin edges represents a graph of the spherical monohedral triangular tiling $TF_{48}$. Cf. [16, Figure 88]. In $gK_{24}$, the inner angles between thick edges (thick edge and dotted edge, dotted edges, resp.) are $\beta = 2B$ ($\alpha = A$, $\delta = 2C$, resp.). The right is the view of $gK_{24}$ from a two-fold axis of rotation. See Subsection 2.2.
Lemma 2. Let $n$ be an integer greater than or equal to three. Then

1. $TR_2^n$ exists and the tile is a rhombus $\iff \alpha = 1 - \frac{1}{n}$. \hspace{1cm} (1)

2. $TR_2^n$ exists and the tile is a dart $\iff \frac{1}{2n} < \alpha < \frac{1}{2}$. \hspace{1cm} (2)

3. $TR_2^n$ exists and the tile is a kite $\iff \frac{1}{2} < \alpha < 1 - \frac{1}{2n}$ and $\alpha \neq 1 - \frac{1}{n}$. \hspace{1cm} (3)

Proof. First we prove that $TR_2^n$ exists as a spherical monohedral quadrangular tiling if and only if $1/(2n) < \alpha < 1 - 1/(2n)$ and $\alpha \neq 1/2$. Let $\mathcal{T}$ be obtained from $TR_2^n$ by drawing the meridian diagonal segment of each quadrangular tile $Q$ of $TR_2^n$. Then $TR_2^n$ exists as a spherical monohedral quadrangular tiling, if and only if $\mathcal{T}$ exists as a spherical monohedral triangular tiling. By Ueno-Agaoka’s classification ([16, Table], included as Table 2) of all spherical monohedral triangular tilings and [16, Figure 4], the tiling $\mathcal{T}$ exists as a spherical monohedral triangular tiling, if and only if $\mathcal{T}$ is a spherical monohedral triangular tiling $G_4^\alpha$ of Table 2 such that the list of inner angles of the triangular tile of $G_4^\alpha$ is $(A, B, C) = (\alpha, \delta/2, \beta/2)$, if and only if $1/(2n) < \alpha < 1 - 1/(2n)$. Thus we have the desired consequence.

The equivalence (1) follows because $\beta = \delta$ if and only if the tile is a rhombus. The equivalence (3) follows because the tile is a kite if and only if all the inner angles $\alpha$, $\beta$ and $\delta$ are less than 1 and $\alpha \neq \delta$. The equivalence (2) follows because $\delta = 2 - 2\alpha > 1$ if and only if the tile is a dart.

Fact 1. $TR_6^{2/3}$ is the central projection of the regular hexahedron to the sphere.
Proof. In $TR_6^{2/3}$, the four inner angles of a tile $Q$ are equal, and the horizontal diagonal segment of a tile and the vertical diagonal segment of the tile both bisect the inner angles of $Q$. Hence, the horizontal diagonal segment of $Q$ subdivides $Q$ into two congruent isosceles triangles $T$ and the vertical diagonal segment of $Q$ subdivides the tile into two isosceles triangles congruent to $T$. Thus the vertical diagonal segment and the horizontal diagonal segment are of the same length. Therefore $Q$ is a regular square. □

2.4. $R_{12}$ and $R_{30}$. A spherical monohedral rhombus-faced tiling $R_{12}$ ($R_{30}$, resp.) is constructed from the spherical monohedral triangular tiling $F_{24}$ ($F_{60}$, resp.) mentioned in the beginning of this Section, by deleting the base edge of each isosceles triangular tile. Roughly speaking, $R_{12}$ ($R_{30}$, resp.) is obtained from $O_8$ ($H_{20}$, resp.) by replacing each edge with a rhombus.

2.5. $gTR_{4n+2}$ ($n \geq 2$). We construct a spherical monohedral rhombus-faced tiling $gTR_{4n+2}$ ($n \geq 2$), as follows: Consider $TR_{4n+2}^{2n/(2n+1)}$ (white edges, the top left image of Figure 4), and choose a hemisphere $\hat{\mathcal{W}}$ (“western hemisphere”) such that on the boundary of the hemisphere $\hat{\mathcal{W}}$, from the north pole $N$, we see a meridian edge, a meridian diagonal segment, another meridian edge and another meridian diagonal segment of $TR_{4n+2}^{2n/(2n+1)}$ (the second image of Figure 4). Every such choice results in a congruent hemisphere, because the polar axis of the tiling $TR_{4n+2}^{2n/(2n+1)}$ is a $(2n+1)$-fold axis of rotation. Consider the copy of the hemisphere $\hat{\mathcal{W}}$, rotate it around the polar axis by $\pi$ radian (the third image of Figure 4). Let the resulting hemisphere be $\hat{\mathcal{W}}'$.

Let $\eta$ be the horizontal axis which passes through the center of the boundary of $\hat{\mathcal{W}}'$ and is normal to the plane spanned by the boundary of $\hat{\mathcal{W}}'$. Rotate $\hat{\mathcal{W}}'$ around the axis $\eta$ in the length of a meridian edge of $TR_{4n+2}^{2n/(2n+1)}$. Let $\mathcal{E}$ be the resulting hemisphere (“eastern hemisphere”). On the boundary of $\mathcal{E}$, we see meridian edges, and meridian diagonal segments, alternatingly. So the boundary of $\mathcal{E}$ corresponds to that of $\hat{\mathcal{W}}'$. Hence gluing $\hat{\mathcal{W}}'$ to $\mathcal{E}$ produces a spherical monohedral rhombus-faced tiling. Let the resulting tiling be $gTR_{4n+2}$ (the white edges of the fourth image of Figure 4).

$gTR_{4n+2}$ with $n = 1$ is $TR_6^{2/3}$ (cf. Fact 1).

Remark 1. The $K_{24}$, $K_{60}$, $R_{12}$ and $R_{30}$ we have constructed are exactly the quadrangle-faced Archimedean duals, topologically. In the terminology of [4, Chapter 21], the corresponding quadrangle-faced Archimedean duals are the tetragonal icosikaitetrahedron, the tetragonal hexacontahedron, the rhombic dodecahedron and the rhombic triacontahedron. The last two Archimedean duals are studied deeply in [6, Sect. 4.7].

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The spherical monohedral kite-faced tiling \( gK_{24} \) has the same skeleton as the dual of the elongated square gyrobicupola (or pseudohombicuboctahedron). The pseudohombicuboctahedron is the 37-th Johnson solids \([13], [18]\) consisting of the eight regular triangles and the 18 regular squares. The vertex figures of this solid are the same, but the action of the symmetry group of the solid on the vertices is not transitive \([14, p. 114]\)\(^1\).

\( gTR_{4n+2} \) \((n \geq 2)\) appears in \([16, Figure 15]\). \( gTR_{10} \) is the rightmost figure of the first line of \([16, Figure 15]\).

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\(^1\) For every Archimedean solid, the faces are regular and the symmetry group acts transitively on the vertices. Prof. M. Deza let the second author know that a nickname of the pseudohombicuboctahedron is the “14-th” Archimedean solid and he suggested that the nickname is due to Grünbaum \([9]\).
3. Classification of spherical monohedral (kite/dart/rhombus)-faced tilings

We say a vertex of the tiling has type \( n_1 \alpha + n_2 \beta + n_3 \delta \), if around the vertex there are \( n_1 \) copies of the inner angle \( \alpha \), \( n_2 \) copies of the inner angle \( \beta \), and \( n_3 \) copies of the inner angle \( \delta \).

**Theorem 2.** (1) The first column of Table 1 is the list of spherical monohedral kite-faced tilings, spherical monohedral dart-faced tilings, and spherical monohedral rhombus-faced tilings.

(2) The non-isohedral tilings in Table 1 are \( gK_{24} \) and \( gTR_{4n+2} \) \((n \geq 2)\).

For each spherical tiling listed in the first column of Table 1, the cyclic list of inner angles of the tile, the symmetry (Schönflies symbol) and the vertex types of the tiling are described in the second, the third, and the fourth columns, respectively.

(3) The tilings appearing in Table 1 are non-congruent to each other.

<table>
<thead>
<tr>
<th>Name of spherical tiling</th>
<th>The cyclic list ((\alpha, \beta, \gamma)) of inner angles of the tile ([\pi \text{ radian}])</th>
<th>Symmetry (Schönflies symbol)</th>
<th>Vertex types ([\text{the number}])</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_{24} )</td>
<td>( x = \delta = \frac{1}{2}, \beta = \frac{3}{7} ) &lt;br&gt;( x = \delta = \frac{1}{2}, \beta = \frac{3}{7} ) &lt;br&gt;( x = \frac{1}{2}, \beta = \frac{2}{7}, \delta = \frac{2}{5} ) &lt;br&gt;( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( O_h ) &lt;br&gt;( D_{4d} ) &lt;br&gt;( I_h ) &lt;br&gt;( D_{4nd} )</td>
<td>( 4\alpha [12], 4\delta [6], 3\beta [8] ) &lt;br&gt;( 4\alpha [8], 3\beta [8], 2\alpha + 2\delta [8], 4\delta [2] ) &lt;br&gt;( 4\alpha [30], 3\beta [20], 5\delta [12] ) &lt;br&gt;( 2\alpha + \delta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( gK_{24} )</td>
<td>( x = \delta = \frac{1}{2}, \beta = \frac{2}{7} ) &lt;br&gt;( x = \frac{1}{2}, \beta = \frac{2}{7}, \delta = \frac{2}{5} ) &lt;br&gt;( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( O_h ) &lt;br&gt;( I_h )</td>
<td>( 4\alpha [6], 3\beta [8] ) &lt;br&gt;( 3\beta [20], 5\alpha [12] )</td>
</tr>
<tr>
<td>( K_{60} )</td>
<td>( x = \frac{1}{2}, \beta = \frac{2}{7}, \delta = \frac{2}{5} ) &lt;br&gt;( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( D_{4nd} )</td>
<td>( 2\alpha + \delta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( TR_{2n} (\frac{1}{2} &lt; x &lt; 1 - \frac{1}{2n}) ) ( (n \geq 3) )</td>
<td>( x = \delta = \frac{3}{7} ) &lt;br&gt;( x = \frac{1}{2}, \beta = \frac{2}{7}, \delta = \frac{2}{5} ) &lt;br&gt;( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( D_{4hd} )</td>
<td>( 2\alpha + \delta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( R_{12} )</td>
<td>( x = \frac{1}{2}, \beta = \frac{2}{7} ) &lt;br&gt;( x = \frac{1}{2}, \beta = \frac{2}{7} ) &lt;br&gt;( { x = \frac{n+1}{n}, \beta = \frac{3}{n} } ) &lt;br&gt;( { x = \frac{2n+1}{2n}, \beta = \frac{2}{2n+1} } )</td>
<td>( O_h ) &lt;br&gt;( I_h ) &lt;br&gt;( D_{4hd} ) &lt;br&gt;( D_{4nd} )</td>
<td>( 4\alpha [6], 3\beta [8] ) &lt;br&gt;( 3\beta [20], 5\alpha [12] ) &lt;br&gt;( 2\alpha + \beta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( R_{30} )</td>
<td>( x = \frac{1}{2}, \beta = \frac{2}{7} ) &lt;br&gt;( x = \frac{1}{2}, \beta = \frac{2}{7} ) &lt;br&gt;( { x = \frac{n+1}{n}, \beta = \frac{3}{n} } ) &lt;br&gt;( { x = \frac{2n+1}{2n}, \beta = \frac{2}{2n+1} } )</td>
<td>( O_h ) &lt;br&gt;( I_h ) &lt;br&gt;( D_{4hd} ) &lt;br&gt;( D_{4nd} )</td>
<td>( 4\alpha [6], 3\beta [8] ) &lt;br&gt;( 3\beta [20], 5\alpha [12] ) &lt;br&gt;( 2\alpha + \beta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( TR_{2n}^{1-1/n} (n \geq 3) )</td>
<td>( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( D_{4hd} )</td>
<td>( 2\alpha + \delta [2n], n\beta [2] )</td>
</tr>
<tr>
<td>( gTR_{4n+2} (n \geq 2) )</td>
<td>( \beta = \frac{2}{7}, 2\alpha + \delta = 2 )</td>
<td>( D_{4hd} )</td>
<td>( 2\alpha + \delta [2n], n\beta [2] )</td>
</tr>
</tbody>
</table>

Table 1. A complete table of the spherical monohedral tilings consisting of (kites/darts/rhombi) (from top to bottom). The three parts are separated by double horizontal lines. The subscript of each tiling indicates the number of tiles. The spherical monohedral kite-faced tiling \( TR_{2n}^{1-1/n} \) and the spherical monohedral dart-faced tiling \( TR_{2n} \) are continuously deformable, and the tilings \( gK_{24} \) and \( gTR_{4n+2} \) are non-isohedral. For the Schönflies symbols \( D_2 \) \((D_{4hd}, O_h, I_h, \text{resp.})\), see pp. 41–42 \( (p. 43, p. 48, p. 49, \text{resp.}) \) in [5]. The tilings appearing in the table are non-congruent to each other. See Theorem 2.
The rest of this section is devoted to the proof of Theorem 2.

As for Theorem 2 (1), the first column of Table 1 are exactly spherical monohedral (kite/dart/rhombus)-faced tilings which are constructed in Section 2. To show that Table 1 is the list of spherical monohedral (kite/dart/rhombus)-faced tilings, we combine Lemma 3, with Ueno-Agaoka’s complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings.

**Definition 2.** We say a spherical monohedral triangular tiling \( T \) **yields** a spherical monohedral quadrangular tiling \( Q \), if \( T \) has twice as many tiles as \( Q \), and the deletion of some edges of \( T \) results in \( Q \).

Then we have the following:

**Lemma 3.** (1) For every spherical tiling \( Q \) by congruent kites or by congruent darts, there is a spherical tiling \( T \) by \( F \) congruent non-equilateral isosceles triangles or by \( F \) congruent scalene triangles, such that \( T \) yields \( Q \) and \( F \geq 12 \).

(2) For every spherical monohedral rhombus-faced tiling \( R \), there is some spherical tiling \( T \) by \( F \geq 12 \) congruent isosceles triangles \( T \), such that \( T \) yields \( R \) and the two base inner angles of the triangle \( T \) is smaller than the other inner angle of \( T \).

**Proof.** The number \( F \) of tiles of \( T \) is an even number greater than or equal to 12, because \( Q \) has more than six tiles by Euler’s theorem (see the first part of the proof of [15, Proposition 1]).

(1) Each tile \( Q \) of the tiling \( Q \) has a vertex \( u \) incident to two edges of length \( a \) and a vertex \( v \) incident to two edges of length \( b \) such that \( a \neq b \), because the tile \( Q \) is a kite or a dart. The diagonal segment between \( u \) and \( v \) subdivides \( Q \) into two congruent triangles \( T \). \( T \) is a non-equilateral isosceles triangle or a scalene triangle, because \( T \) has an edge of length \( a \) and an edge of length \( b \) such that \( a \neq b \). Hence, subdividing each tile \( Q \) of the tiling \( Q \) results in a spherical tiling by congruent non-equilateral isosceles triangles \( T \) or by congruent scalene triangles \( T \). (2) Assume otherwise. Then there is a spherical monohedral rhombus-faced tiling \( R \) with the following condition: whenever a spherical tiling \( T \) by congruent isosceles triangles \( T \) yields \( R \), the two base angles of the isosceles triangle \( T \) is greater than or equal to the other inner angle. Now let the cyclic list of four inner angles of the rhombic tile \( R \) be \( (x, \beta, x, \beta) \). A subdivision of \( R \) results in an isosceles triangle such that the list of inner angles is \( (x, \beta/2, \beta/2) \) or \( (\beta, x/2, x/2) \). Moreover, \( \beta/2 \geq x \) and \( x/2 \geq \beta \). This implies \( x = \beta = 0 \), which is a contradiction.
Table 2. The classification of the spherical monohedral triangular tilings [16, Table]. The first column lists the names. The explicit constructions of the tilings are explained in [16, Section 2]. There are ten sporadic tilings (above the horizontal line) and ten series (below the horizontal line). The mark □ indicates that the tiling is continuously deformable. None of these tilings are isomorphic to each other except the trivial case given by the exchange of $A + B + C = 2$ ($F_4$), $A + B = 1$ ($G_{4n}, TG_{5n}C$).

<table>
<thead>
<tr>
<th>Name</th>
<th>$V$</th>
<th>$E$</th>
<th>inner angles $A, B, C \ [\pi \text{ rad}]$</th>
<th>vertex type $[\text{number}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>4</td>
<td>6</td>
<td>$A + B + C = 2, \ A + B + C = 1/n$</td>
<td>$A + B + C \ [4]$</td>
</tr>
<tr>
<td>$F_{12}^I$</td>
<td>8</td>
<td>18</td>
<td>$A = 2/3, B = C = 1/3$</td>
<td>$3A \ [4], 6B \ [4]$</td>
</tr>
<tr>
<td>$F_{12}^H$</td>
<td>8</td>
<td>18</td>
<td>$A = 2/3, B = C = 1/3$</td>
<td>$A + 4B \ [2], 6B \ [2]$</td>
</tr>
<tr>
<td>$F_{12}^{HII}$</td>
<td>8</td>
<td>18</td>
<td>$A = 2/3, B = C = 1/3$</td>
<td>$A + 4B \ [3], 6B \ [1]$</td>
</tr>
<tr>
<td>$F_{24}$</td>
<td>14</td>
<td>36</td>
<td>$A = 2/3, B = C = 1/4$</td>
<td>$3A \ [8], 8B \ [6]$</td>
</tr>
<tr>
<td>$F_{48}$</td>
<td>26</td>
<td>72</td>
<td>$A = 1/2, B = 1/3, C = 1/4$</td>
<td>$4A \ [12], 6B \ [8]$</td>
</tr>
<tr>
<td>$TF_{48}$</td>
<td>26</td>
<td>72</td>
<td>$A = 1/2, B = 1/3, C = 1/4$</td>
<td>$4A \ [8], 6B \ [8]$</td>
</tr>
<tr>
<td>$F_{60}^I$</td>
<td>32</td>
<td>90</td>
<td>$A = 2/3, B = C = 1/5$</td>
<td>$3A \ [20], 10B \ [12]$</td>
</tr>
<tr>
<td>$F_{60}^H$</td>
<td>32</td>
<td>90</td>
<td>$A = 2/5, B = C = 1/3$</td>
<td>$5A \ [12], 6B \ [20]$</td>
</tr>
<tr>
<td>$F_{120}$</td>
<td>62</td>
<td>180</td>
<td>$A = 1/2, B = 1/3, C = 1/5$</td>
<td>$4A \ [30], 6B \ [20]$</td>
</tr>
<tr>
<td>$TG_{5n}$</td>
<td>4n</td>
<td>12n</td>
<td>$A = 1/2, C = 1/n$</td>
<td>$2.4 + 2B \ [4n]$</td>
</tr>
<tr>
<td>$TG_{5n+4}$</td>
<td>4n+4</td>
<td>12n+6</td>
<td>$A = 1/2, C = 1/n$</td>
<td>$2.4 + 2B \ [4n]$</td>
</tr>
<tr>
<td>$MTG_{5n+4}^I$</td>
<td>4n+4</td>
<td>12n+6</td>
<td>$A = 1/2, C = 1/n$</td>
<td>$2.4 + 2B \ [4n]$</td>
</tr>
<tr>
<td>$MTG_{5n+4}^H$</td>
<td>4n+4</td>
<td>12n+6</td>
<td>$A = 1/2, C = 1/n$</td>
<td>$2.4 + 2B \ [4n]$</td>
</tr>
<tr>
<td>$H_{4n}$</td>
<td>2n+2</td>
<td>6n</td>
<td>$A = 2/3, B = C = 1/n$</td>
<td>$4A + C \ [2n], 6B \ [4n]$</td>
</tr>
<tr>
<td>$TH_{5n+4}$</td>
<td>4n+4</td>
<td>12n+6</td>
<td>$A = 2/3, B = C = 1/n$</td>
<td>$4A + C \ [4n]$</td>
</tr>
<tr>
<td>$I_{5n}$</td>
<td>4n+2</td>
<td>12n</td>
<td>$A = 1/2, B = 2/3, C = 1/n$</td>
<td>$4B + 2C \ [2n]$</td>
</tr>
<tr>
<td>$TI_{5n+8}$</td>
<td>8n+6</td>
<td>24n+12</td>
<td>$A = 1/2, B = 2/3, C = 1/n$</td>
<td>$4B + 2C \ [4n]$</td>
</tr>
</tbody>
</table>

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We will prove all spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tilings appear in Table 1, by exhausting the pairs \((\mathcal{T}, \mathcal{Q})\) of Lemma 3 (1) (Lemma 3 (2), resp.) in Lemma 5 (Lemma 7, resp.), with Lemma 4 (Lemma 4 (1), resp.).

**Lemma 4.** Let \(F \geq 12\) be an even number. Let \(\mathcal{T}\) be a spherical tiling by \(F\) congruent isosceles triangles \(T\). In Ueno-Agaoka’s complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings, the following two assertions hold: Inside “\(\{\cdots\}\)”, the greatest and the smallest inner angles of the triangular tile \(T\) are given.

(I) If the two base angles of the spherical isosceles triangle \(T\) are smaller than the other inner angle of \(T\), then the spherical monohedral triangular tiling \(T\) is one the following:
- \(F_{12}I, F_{12}II, F_{24}, F_{60}, F_{60}II\),
- \(G_{4n}\ (n \geq 3) \{1 - 1/n, 1/n\}\),
- \(TG_{8n}\ (n \geq 1) \{2n/(2n + 1), 1/(2n + 1)\}\),
- \(MTG_{12}I\ \{2/3, 1/3\}\),
- \(I_{24}\ \{1/2, 1/3\}\),
- \(H_{12}\ \{2/3, 1/3\}, H_{16}\ \{1/2, 3/8\}\).

(II) If the two base angles of the spherical isosceles triangle \(T\) are greater than the other inner angle, then the spherical monohedral triangular tiling \(T\) is one of the following:
- \(G_{4n+2}\ (n \geq 3) \{1/2, 2/(2n + 1)\}\),
- \(TG_{8n}\ (n \geq 2) \{1/2, 1/(2n)\}\),
- \(H_{4n}\ (n \geq 6), TH_{8n+4}\ (n \geq 3)\),
- \(G_{4n}\ (n \geq 3) \{1/2, 1/n\}\),
- \(TG_{8n+4}\ (n \geq 1) \{1/2, 1/(2n + 1)\}\).

**Proof.** The lemma is [16, Proposition 6] where the number \(F\) is greater than or equal to 12.

In Subsubsection 3.1.2, to prove Theorem 2 (2), we will use the following:

**Fact 2.** Let \(\mathcal{T}\) be a spherical tiling by congruent triangles such that the list of three inner angles is \((A, B, C)\), and let \(\mathcal{Q}\) be a spherical monohedral (kite/dart)-faced (rhombus-faced, resp.) tiling such that the cyclic list of four inner angles of the tile is \((x, \beta, x, \delta)\) \((A, \beta, \alpha, B), \text{ resp.}\).

If \(\mathcal{T}\) yields \(\mathcal{Q}\) and \((x, \beta, x, \delta) = (A, 2B, A, 2C)\) \((A, \beta, x, B) = (2B, A, 2B, A), \text{ resp.}\), then the vertex types of \(\mathcal{Q}\) are exactly obtained from those of \(\mathcal{T}\) by applying a substitution \((A := x, B := \beta/2, C = \delta/2)\) \((A := \beta, B := x/2), \text{ resp.}\).
Theorem 2 (3) follows from Theorem 2 (2), because spherical monohedral tilings in Table 1 are not congruent to each other by the form of the tiles (i.e., kites, darts, rhombi), by the symmetry, or by the number of the tiles.

3.1. A complete classification of spherical monohedral (kite/dart)-faced tilings.

3.1.1. The completeness (Theorem 2 (1)).

Lemma 5. Let $\mathcal{T}$ be a spherical tiling by congruent non-equilateral triangles $T$, and $\mathcal{Q}$ be a spherical kite-faced or dart-faced monohedral tiling. Then if $\mathcal{T}$ yields $\mathcal{Q}$, then $(\mathcal{T}, \mathcal{Q})$ is $(F_{48}, K_{24})$, $(TF_{48}, gK_{24})$, $(F_{120}, K_{60})$, or $(G_{ab}, TR_{2n})$ for some integer $n \geq 3$ and some $a \in (1/2n, 1 - 1/(2n)) \setminus \{1/2, 1 - 1/n\}$.

To prove the previous lemma, we will use the following “uniqueness” lemma:

Lemma 6. Assume that $\mathcal{T}$ is a spherical tiling by congruent triangles $T$, $\mathcal{Q}$ is a spherical tiling by congruent quadrangles $Q$, and $\mathcal{T}$ yields $\mathcal{Q}$ in a manner that $Q$ is a kite or a dart. Then the following two assertions hold:

1. $\mathcal{T}$ is none of entries $TG_{8n+4}$ $(n \geq 1)$, $MTG_{8n+4}^I$ $(n \geq 2)$, and $MTG_{8n+4}^H$ $(n \geq 2)$ of Table 2.

2. When $T$ is a non-equilateral isosceles triangle, $T$ is not rectangular. Moreover, the quadrangular tile $Q$ is unique modulo congruence for the given spherical monohedral triangular tiling $\mathcal{T}$.

Proof. From the assumption, $\mathcal{T}$ yields $\mathcal{Q}$ in a manner that $Q$ is a kite or a dart. So, the tile $Q$ of $\mathcal{Q}$ has a pair $(2X, 2Y)$ of opposite inner angles for some inner angles $X, Y$ of the triangular tile $T$. It is because we must glue two copies of the triangular tile $T$ of $\mathcal{T}$ at the edges of the same length, to obtain the quadrangular tile $Q$ of $\mathcal{Q}$. Thus there is an inner angle $X$ of $T$ such that the number of occurrences of $X$ is even for any vertex type of $\mathcal{T}$.

According to Table 2, however, if $\mathcal{T}$ is $TG_{8n+4}$ $(n \geq 1)$ or $MTG_{8n+4}^I$ $(n \geq 2)$, then for every inner angle $X \in \{A, B, C\}$, the number of occurrences of $X$ is odd in a vertex type $A + B + (2n + 1)C$ of $\mathcal{T}$. So $TG_{8n+4}$ $(n \geq 1)$ and $MTG_{8n+4}^I$ $(n \geq 2)$ are impossible. Similarly, for $MTG_{8n+4}^H$ $(n \geq 2)$, for every $X \in \{A, B, C\}$, the number of occurrences of $X$ is odd in a vertex type $A + 3B + C$ of $MTG_{8n+4}^H$. So $MTG_{8n+4}^H$ $(n \geq 2)$ is impossible too. Thus the assertion (1) follows.

The first part of the assertion (2) is proved as follows. The list of three inner angles of the isosceles triangular tile $T$ is $(X, X, Y)$ without loss of
generality. Because the quadrangular tile $Q$ is not a rhombus, the cyclic list of the four inner angles is $(X, 2Y, X, 2Y)$. If $X$ or $Y$ is $\pi/2$ radian, then some inner angle of $Q$ is $\pi$ radian, which is a contradiction. Hence, the triangular tile of $\mathcal{T}$ is not rectangular. This argument proves the second part of the assertion (2), as well. This completes the proof of the assertion (2) and thus Lemma 6.

**Proof of Lemma 5.** We use Lemma 3 (1). The non-equilateral triangular tile $T$ is either a scalene triangle (Case 1) or a non-equilateral isosceles triangle (Case 2).

Case 1. The tile $T$ is a scalene triangle.

By Lemma 6 (1), the spherical tiling $\mathcal{T}$ by $F \geq 12$ congruent scalene triangles is one of the following entries of Table 2 (Ueno-Agaoka’s complete classification [16, Table] of the spherical monohedral triangular tilings):

$$F_{48}, TF_{48}, F_{120}, G_{4n} \ (n \geq 3), \ I_{8n} \ (n \geq 4), \text{ or } TI_{16n+8} \ (n \geq 2).$$

The case $\mathcal{T} = I_{8n} \ (n \geq 4)$ or $TI_{16n+8} \ (n \geq 2)$ is impossible, by Table 2. Indeed, $\mathcal{T}$ has a vertex of type $4A$ and $4B + 2C$. Here $(A, B, C)$ is the list of three inner angles of the triangular tile of $\mathcal{T}$. By $A = 1/2$ and Proposition 1, the cyclic list of four inner angles of the tile $Q$ of $\mathcal{D}$ is $(\alpha, \beta, \alpha, \delta) = (A, 2B, A, 2C)$. Hence, some $\mathcal{T}$-vertex of type $4B + 2C$ causes some $\mathcal{D}$-vertex $v$ of type $2\beta + \delta$. So the $\mathcal{T}$-edge $CA$ and the $\mathcal{T}$-edge $BA$ are both $\mathcal{D}$-edges and should match, but this is impossible because the triangular tile $ABC$ of $\mathcal{T}$ is scalene.

Therefore the remaining cases are $\mathcal{T} = F_{48}, TF_{48}, F_{120}, G_{4n} \ (n \geq 3)$.

Case 1.1. $\mathcal{T} = F_{48}, F_{120}$ or $TF_{48}$.

If a spherical tiling $\mathcal{T} = F_{48}$ (resp.) by congruent right scalene triangles yields a spherical monohedral quadrangular tiling $\mathcal{D}$, then $\mathcal{D}$ is $K_{24}$ ($K_{60}$, resp.), by the construction (Subsection 2.1) of $K_{24}$ ($K_{60}$, resp.) and the following:

**Fact 3.** Every spherical tiling $\mathcal{T}$ by congruent right scalene triangles $T$ yields at most one spherical monohedral quadrangular tiling $\mathcal{D}$, modulo congruence.

**Proof.** By Proposition 1, the tile $Q$ of $\mathcal{D}$ cannot be a parallelogram. If $\mathcal{T}$ yields another spherical monohedral quadrangular tiling, we must remove a cathetus of $T$, so some inner angle of $Q$ should be $1$. This is a contradiction.

A spherical monohedral triangular tiling $TF_{48}$ yields $gK_{24}$. It is proved as follows: For each tile of $gK_{24}$, draw a diagonal segment between opposite
vertices having unequal inner angles. This subdivision of the tiles results in a spherical tiling \( \mathcal{Q}' \) consisting of 48 congruent right scalene triangles having inner angles \( 1/3 \) and \( 1/4 \). It is because \( K_{24} \) has the same tile by the construction of \( gK_{24} \). According to Table 2, \( \mathcal{Q}' \) is either \( F_{48} \) or \( TF_{48} \). \( F_{48} \) yields \( K_{24} \) but not \( gK_{24} \) by Fact 3. So \( TF_{48} \) yields \( gK_{24} \).

Case 1.2. \( \mathcal{Q} = G_{4n} \) \( (n \geq 3) \).

Let the three inner angles \( A, B, C \) of the triangular tile \( T \) of \( \mathcal{Q} = G_{4n} \) satisfy that \( A + B = 1, C = 1/n, \) and \( 1/(2n) < A, B < (2n - 1)/(2n) \) as in Table 2. Since the triangular tile \( T \) is scalene, none of \( A \) and \( B \) is \( C = 1/n \), and we can assume \( A > B \), without loss of generality. Hence

\[
1 - \frac{1}{2n} > 1 - B = A > \frac{1}{2} > B > \frac{1}{2n} \quad \text{and} \quad B \neq \frac{1}{n},
\]

The Mollweide-like projection of \( G_{4n} \) is given by the thick edges and the thin edges of Figure 5, according to [16, Figure 4, p. 471]. Let the spherical monohedral triangular tiling \( \mathcal{Q} = G_{4n} \) yield a tiling \( \mathcal{Q} \) consisting of \( 2n \) congruent quadrangles \( Q \). If \( Q \) is obtained by gluing two copies of the triangular

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Fig. 5. The set of the thick edges and thin edges represents the Mollweide-like projection of \( G_{4n} \), according to [16, Figure 4, p. 471]. The marks \( o, \bullet \) in the figure stand for the angles \( A, C \), while the other inner angles are \( B \). Identify the leftmost thin (thick, resp.) edge with the rightmost thin (thick, resp.) edge. The tiling \( TR_{2n}^{A} \) by congruent kites is determined by the set of the thin edges \( (NA's \ and \ SA's) \) and the horizontal thick edges \( (AB's) \), and the tiling \( TR_{2n}^{B} \) by congruent darts is determined by the thick edges \( (NB's, SB's, \ and \ AB's) \). Here we suppose the inner angles satisfy \( A + B = 1, C = 1/n, A > B \). See Case 1.2.
tile $T$ of $\mathcal{T}$ at their edges $AB$, then the tile $Q$ becomes a lune (i.e., a digon) but not a quadrangle because $A + B = 1$. Hence the tile $Q$ is obtained either by gluing two copies of tile $T$ (Case (i)) at the edges $AC$ (thin edges in Figure 5) or (Case (ii)) at the edges $BC$.

Case (i). The graph of $\mathcal{T}$ is given by the thick edges of Figure 5, and is of the form Figure 3 (right). The cyclic list of four inner angles of $Q$ is $(B, 2A, B, 2C)$. The tile $Q$ is a dart, because Lemma 2 (2) and the above (4). Thus the tiling $\mathcal{T}$ is listed as $TR_{2n}^d (1/(2n) < B < 1/2, B \neq 1/n)$ in the fifth entry of Table 1 with $(\alpha, \beta, \gamma, \delta) = (B, 2C, B, 2A)$.

Case (ii). The graph of $\mathcal{T}$ is given by the thin edges of Figure 5 and the horizontal thick edges (the “equator.”) The cyclic list of four inner angles of $Q$ is $(A, 2B, A, 2C)$. By above (4), the angle $A$ ranges over $(1/2, 1 - 1/(2n)) \setminus \{1 - 1/n\}$. By Lemma 2 (3), the tiling $\mathcal{T}$ is $TR_{2n}^d$ of the form Figure 3 (left).

Applying the same argument of Case (i) and Case (ii) to the case $A < B$, we obtain $TR_{2n}^d$ where $A \in (1/(2n), 1 - 1/(2n)) \setminus \{1/2, 1 - 1/n, 1/n\}$.

Case 2. $\mathcal{T}$ is a spherical tiling by $F \geq 12$ congruent non-equilateral isosceles triangles $T$.

Let $B$ and $C$ be the two base angles of $T$ and $A (\neq B = C)$ be the other inner angle of the triangular tile $T$. If a kite or a dart is obtained from gluing two copies of the isosceles triangle $T$, it has inner angle $2A$. Actually, the cyclic list of the four inner angles is $(2A, B, 2C, B)$, $(2A, C, 2B, C)$, or $(2A, B, C + B, C)$. Thus $\mathcal{T}$ cannot have a vertex type where $A$ occurs odd times. So $\mathcal{T}$ is none of $F_{12}^I$, $F_{12}^II$, $F_{12}^{III}$, $F_{24}$, $F_{60}I$, $F_{60}II$, $H_{4n}$ $(5 \neq n \geq 3)$ and $TH_{8n+4}$ $(n \geq 3)$. (In Table 2, the non-base inner angle of the non-equilateral isosceles triangular tile of $H_{4n}$ $(5 \neq n \geq 3)$ $(TH_{8n+4} (n \geq 3))$ is $C$.) Thus if $B = C > A$, then no tiling is possible, by Lemma 4 (II) and Lemma 6 (2). Hence, the tiling $\mathcal{T}$ amounts to be $G_{4n}$ $(n \geq 3)$ where $A = 1 - 1/n$ and $B = C = 1/n$, by the previous two sentences, Lemma 4 (I), and Lemma 6 (1), since Lemma 6 (2) implies $\mathcal{T} \neq I_{2n}$.

From [16, Figure 11], we observe that the Mollweide-like projection of $G_{4n}$ $(A = 1 - 1/n > B = C = 1/n, n \geq 3)$ is given by the thick edges ($NC’s$, $SC’s$, and $AC’s$) and the thin edges ($NA’s$ and $SA’s$) of Figure 6. Here the upper leftmost thin (thick, resp.) edge is identified with the upper rightmost thin (thick, resp.) edge. In Figure 6, the Mollweide-like projection of the spherical monohedral dart-faced tiling $\mathcal{T}$ is determined by the thick edges ($NC’s$, $SC’s$, and $AC’s$). Note that the set of the thin edges ($NA’s$ and $SA’s$) and the horizontal thick edges ($CA’s$) determines the Mollweide-like projection of a spherical monohedral rhombus-faced tiling by $B = C$.

The cyclic list $(\alpha, \beta, \gamma, \delta)$ of inner angles of the quadrangular tile of $\mathcal{T}$ is $(C, 2B, C, 2A) = (1/n, 2/n, 1/n, 2 - 2/n)$. So the tiling is indeed a dart-
faced \( TR_{2n} \) by Lemma 2 (2) and \( n \geq 3 \). This completes the proof of Lemma 5.

By Lemma 5 and Lemma 3 (1), we have verified Theorem 2 (1) for the case the tile is a kite or a dart.

3.1.2. **Theorem** 2 (2). By using a procedure \([5, p. 55 and Figure 3.10 (p. 56)]\), we will prove Theorem 2 (2) for spherical monohedral kite-faced tilings (1) \( K_{24} \) and \( K_{60} \), (2) \( gK_{24} \) and (3) \( TR_{2n}^2 (1/2 < x < 1 - 1/(2n), \ x \neq 1 - 1/n, \ n \geq 3) \). Then we will prove Theorem 2 (2) for the spherical monohedral dart-faced tiling (4) \( TR_{2n} (1/(2n) < x < 1/2, \ n \geq 3) \).

(1) \( K_{24} \) (\( K_{60}, \) resp.) is isohedral. It is proved by Lemma 1 as follows: Each triangular (pentagonal, resp.) tile \( \Phi \) of \( C_8 (D_{12}, \text{resp.}) \) can be transformed to another tile \( \Phi' \) of \( C_8 (D_{12}, \text{resp.}) \) by the symmetry of \( C_8 (D_{12}, \text{resp.}) \). Within the tile \( \Phi \), any tile of \( K_{24} (K_{60}, \text{resp.}) \) can be transformed to any other tile of \( K_{24} (K_{60}, \text{resp.}) \), due to the following:

**Fact 4.** Let \( P \) be the central projection of a Platonic solid with the faces being \( p \)-gons to the sphere. Then for each tile \( T \) of \( P \), a \( p \)-fold axis of rotation of \( P \) is through the center of \( T \).

**Proof.** See \([5, p. 47]\) for \( P \) being the central projection of the regular tetrahedron, \([5, p. 49]\) for \( P = D_{12} \) and \( H_{20} \), and \([5, p. 48]\) for the other \( P \).
Hence any tile of $K_{24}$ ($K_{60}$, resp.) can be transformed to any other tile of $K_{24}$ ($K_{60}$, resp.), as desired.

The cyclic list of inner angles of the tile and the list of vertex types of $K_{24}$ ($K_{60}$, resp.) are as in Table 1. It is due to Fact 2, since according to Lemma 5, the graph of $K_{24}$ ($K_{60}$, resp.) is obtained from that of $F_{48}$ ($F_{120}$, resp.), by deleting all the hypotenuses of the triangular tiles joining the vertex $B$ and vertex $C$.

(2) The list of inner angles of the tile of $gK_{24}$ is the list of inner angles of the tile of $K_{24}$, by the construction (Subsection 2.2).

The Schönflies symbol of $gK_{24}$ is $D_{4d}$. We prove it according to the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

The polar axis is the only four-fold rotation axis of $gK_{24}$. The polar axis of $gK_{24}$ is a four-fold axis of rotation, because $gK_{24}$ is obtained from $K_{24}$ by gliding the northern hemisphere around the four-fold, polar axis of $K_{24}$. Assume $\rho$ is another four-fold axis of rotation. Then $\rho$ is not through a tile, as the tile is not regular. The four-fold axis $\rho$ of rotation is not through a vertex, because every four-valent vertex other than the pole is incident to edges of different length.

$gK_{24}$ has no three-fold axis of rotation. Assume $\rho$ is a three-fold axis of rotation. Then $\rho$ is through neither a quadrangular tile nor a four-valent vertex. If $\rho$ is through a three-valent vertex, then the antipodal point is strictly between edges (or is a four-valent vertex), according to the image (Figure 2 (right)).

By the same image, the tiling $gK_{24}$ has (i) four two-fold axes of rotation perpendicular to the four-fold axis of rotation, and (ii) four mirror planes passing between two-fold axes of rotation. Each mirror plane contains exactly six consecutive edges. Thus the Schönflies symbol of $gK_{24}$ is $D_{4d}$, as desired.

The tiling $gK_{24}$ is not isohedral, because the number 24 of the tiles does not divide the order 16 of the symmetry group $D_{4d}$.

The cyclic list of inner angles of the tile and the list of vertex types of $gK_{24}$ are as in Table 1. It is due to Fact 2 and Lemma 5, because, as in Figure 2 (left), we can obtain the graph (thick edges and dotted edges) of $gK_{24}$ from the graph (thin edges, thick edges and dotted edges) of $TF_{48}$ by deleting each hypotenuse joining a vertex of the inner angle $B$ and a vertex of the inner angle $C$ ($\neq B$).

(3) In Figure 3 (left), for the cyclic list of inner angles $(\alpha, \beta, \alpha, \delta)$ of $TR_{2n}^3$, we see the type of each pole is $n\beta$ and the type of each non-pole vertex is $2\alpha + \delta$. Hence, the inner angles and the vertex types are as in Table 1.

The tile is a kite, by Lemma 2 (3) and the condition on the inner angle $\alpha$. Thus the length $a$ of each meridian edge is not equal to the length $b$ of each non-meridian edge, in the tiling.
We prove that the Schönflies symbol of $TR_{2n}^z$ is $D_{nd}$, by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If $TR_{2n}^z$ has a three-fold or four-fold axis of rotation passing through a vertex, then the axis is the polar axis. It is verified as follows: Every non-pole vertex is three-valent and incident to an edge of length $a$ and to two edges of length $b$ ($\neq a$). A three-fold axis of rotation is not through an inner point of a tile $T$, because $T$ is a non-regular quadrangle. Obviously, a three-fold or four-fold axis of rotation cannot pass through a point of an edge of $TR_{2n}^z$.

Clearly, $TR_{2n}^z$ has an $n$-fold axis of rotation through the poles.

$TR_{2n}^z$ has $2n$ two-fold axes of rotation, perpendicular to $\rho$. It is because $TR_{2n}^z$ has $2n$ points where the equator and the non-meridian edges intersect, and has $2n$ vertical mirror planes which pass between the horizontal two-fold axes of rotation. So the Schönflies symbol of $TR_{2n}^z$ is $D_{nd}$, as desired.

$TR_{2n}^z$ is isohedral. Indeed, if two quadrangular tiles $T$ and $T'$ of the tiling share a meridian edge of the tiling, then $T$ is rotated to $T'$ around the vertical $n$-fold axis of rotation. If $T$ and $T'$ share a non-meridian edge of the tiling $TR_{2n}^z$, then $T$ is rotated to $T'$ around a horizontal two-fold axis of rotation.

(4) It is proved as in the previous assertion (3).

This completes the proof of Theorem 2 (1) and (2) for the case where the tile is a kite or a dart.

3.2. A complete classification of spherical monohedral rhombus-faced tilings.

3.2.1. The completeness (Theorem 2 (1)).

Lemma 7. Suppose $\mathcal{R}$ is a spherical monohedral rhombus-faced tiling and suppose that $\mathcal{T}$ is a spherical tiling by congruent non-equilateral isosceles triangles $T$ such that the list of inner angles $A, B, C$ of $T$ satisfy $A > B = C$. Then if $\mathcal{T}$ yields $\mathcal{R}$, then one of the following holds:

1. $\mathcal{R}$ is $TR_{6}^{2/3}$ while $\mathcal{T}$ is $F_{12}^I, F_{12}^{II}, F_{12}^{III}, MTG_{12}^I, H_{12}, TG_{12}$ ($A = 2/3, B = C = 1/3)$, or $G_{12}$ ($A = 2/3, B = C = 1/3$);
2. $\mathcal{R}$ is $R_{12}$ while $\mathcal{T}$ is $F_{24}^I$ or $I_{24}$;
3. $\mathcal{R}$ is $R_{30}$ while $\mathcal{T}$ is $F_{60}^I$ or $F_{60}^{II}$;
4. $\mathcal{R}$ is $TR_{8}^{3/4}$ while $\mathcal{T}$ is $H_{16}$ ($A = 1/2, B = C = 3/8$). In the entry of $H_{16}$ of Table 2, we read $A = B = 3/8 < C = 1/2$) or $\mathcal{T}$ is $G_{16}$ ($A = 3/4, B = C = 1/4$);
5. $\mathcal{R}$ is $TR_{2n}^{1-1/n}$ and $\mathcal{T}$ is $G_{4n}$ with $A = 1 - 1/n, B = C = 1/n$ for $n \geq 5$; or
6. $\mathcal{R}$ is $gTR_{4n+2}$ and $\mathcal{T}$ is $TG_{8n+4}$ with $A = 2n/(2n + 1), B = C = 1/(2n + 1)$ for $n \geq 2$. 

□
We will prove the previous lemma by using Lemma 3 (2) and the following “uniqueness” lemma:

**Lemma 8.** Every spherical tiling $T$ by congruent non-equilateral isosceles triangles $T$ yields a spherical monohedral rhombus-faced tiling $R$ uniquely modulo congruence. If the list of inner angles of $T$ is $(A, B, C)$ and $A \neq B = C$, then the cyclic list of inner angles of the rhombic tile of $R$ is $(A, 2B, A, 2C)$.

**Proof.** By removing the base edges of $T$, we obtain a spherical monohedral rhombus-faced tiling. If $T$ yields another spherical monohedral quadrangular tiling, then the cyclic list of inner angles of $Q$ should be $(2B, C, 2A, C)$. Since $A \neq B$, the tile $Q$ is a kite or a dart.

**Proposition 2.** (1) For any $n \geq 3$, the spherical monohedral triangular tiling $G_{4n} (A = 1 - 1/n, B = C = 1/n)$ ($H_{4n}$, resp.) is obtained from the spherical monohedral rhombus-faced tiling $TR_{2n}^{1-1/n}$ by drawing the meridian (the “horizontal,” resp.) diagonal segment of each tile. $I_{8n} = \text{kis}(TR_{2n}^{1-1/n})$ ($n \geq 3$).

(2) The spherical monohedral triangular tiling $TH_{8n+4}$ ($n \geq 3$) is obtained from the spherical monohedral rhombus-faced tiling $gTR_{4n+2}$ by drawing the shorter diagonal segment of each tile. $TI_{16n+8} = \text{kis}(gTR_{4n+2})$ ($n \geq 2$).

**Proof.** (1) See Figure 11, Figure 12, Figure 14, Figure 20, and Figure 21 of [16]. (2) See [16, Figure 21].

**The proof of Lemma 7.** We use Lemma 3 (2). Let $T$ be one of spherical monohedral triangular tilings listed in Lemma 4 (I).

Case 1. $T$ is $F_{12}^I$, $F_{12}^H$, $F_{12}^{III}$, $MTG_{12}^I$, $H_{12}$, $G_{12}$ ($A = 2/3, B = C = 1/3$), or $TG_{12}$ ($A = 2/3, B = C = 1/3$).

We draw one diagonal segment in each square tile of $TR_6^{2/3}$. Then the resulting tiling is exactly a tiling $T$. Combinatorially, there exist just seven tilings, which are exactly the tilings $T$ of this case, according to [16, pp. 481–482]. So $T$ yields $TR_6^{2/3}$.

Case 2. $T$ is $F_{24}$ or $I_{24}$.

By the construction (Subsection 2.4) of $R_{12}$, the spherical tiling $F_{24}$ by congruent right, non-equilateral isosceles triangles yields $R_{12}$.

The spherical tiling $I_{24}$ by congruent non-equilateral isosceles triangles yields $R_{12}$, too. It is verified as follows: According to Subsection 2.4, $R_{12}$ is constructed from $F_{24} = \text{kis}(C_8)$ by deleting the 12 edges of $C_8$. By recalling the image [16, Figure 20] of $R_{12}$ and the duality between $TR_6^{2/3}$ and $C_8$, we can construct $R_{12}$ from $I_{24} = \text{kis}(TR_6^{2/3})$ by deleting the 12 edges of $TR_6^{2/3}$. 
Case 3. $\mathcal{T}$ is $F_{60}^I$ or $F_{60}^{II}$.

By the construction (Subsection 2.4) of $R_{30}$, spherical monohedral triangular tiling $F_{60}^I$ yields $R_{30}$. Moreover, $F_{60}^{II}$ yields $R_{30}$, by [16, Figure 36].

Case 4. $\mathcal{T}$ is $H_{16}$ or $G_{4n}$ ($A = 1 - 1/n, B = C = 1/n, n \geq 4$).

Proposition 2 (1) implies the assertion (4) and the assertion (5) of Lemma 7.

Case 5. $\mathcal{T}$ is $TG_{8n+4}$ ($A = 2n/(2n+1), B = C = 1/(2n+1), n \geq 2$).

Then $\mathcal{T}$ yields $gTR_{4n+2}$. It is proved as follows: subdivide each rhombic tile of $gTR_{4n+2}$ into two congruent non-equilateral isosceles triangles, by drawing the longer diagonal segment. This results in a spherical monohedral tilings $\mathcal{T}$ consisting of $8n+4$ non-equilateral isosceles triangles. The cyclic list of inner angles of the rhombic tile of $gTR_{4n+2}$ and the cyclic list of inner angles of the isosceles triangular tile of $\mathcal{T}$ are $(2n/(2n+1), 1/(2n+1), 1/(2n+1))$.

Owing to Ueno-Agaoka’s classification (Table 2) of all spherical monohedral triangular tilings, the spherical monohedral triangular tiling $\mathcal{T}$ is either $G_{8n+4}$ or $TG_{8n+4}$. Here we have verified that $G_{8n+4}$ yields $TR_{4n+2}^{2n/(2n+1)}$, in the previous Case 4. By Lemma 8, a spherical monohedral, isosceles triangular tiling $TG_{8n+4}$ yields $gTR_{4n+2}$. This completes the proof of Lemma 7.

By Lemma 7 and Lemma 3 (2), we established Theorem 2 (1) for the case the tile is a rhombus.

$\square$

3.2.2. Theorem 2 (2).

Lemma 9. $gTR_{4n+2}$ ($n \geq 2$) is the spherical monohedral, non-isohedral rhombus-faced tiling by the tiles of spherical monohedral tiling $TR_{4n+2}^{2n/(2n+1)}$. The smallest inner angle $\beta$ and the largest inner angle $\alpha$ of the rhombic tile are

$$
\beta = \frac{2}{2n+1}, \quad \alpha = \frac{2n}{2n+1}.
$$

The Schönflies symbol of $gTR_{4n+2}$ is $D_2$. The vertex types of $gTR_{4n+2}$ are as in Table 1.

Actually, the edges of $gTR_{4n+2}$ have length

$$
a = \arccos \left( \cos \frac{\pi \beta}{2} \left/ \left( 1 + \cos \frac{\pi \beta}{2} \right) \right. \right) \neq \frac{\pi}{2}.
$$

The vertices $v_i, u_i$ ($0 \leq i \leq 2n+1$) have the following cartesian coordinates:
\[
v_i = \left( \sin a \cos \frac{i\pi\beta}{2}, -\sin a \sin \frac{i\pi\beta}{2}, (-1)^i \cos a \right).
\]
(6)

\[
u_i = \left( \frac{(-1)^i - \cos \frac{i\pi\beta}{2}}{2}, \sin 2a, \sin a \sin \frac{i\pi\beta}{2}, \sin^2 a \cos \frac{i\pi\beta}{2} + (-1)^i \cos^2 a \right).
\]
(7)

**Proof.** By Lemma 8 and Fact 2, the vertex types of \(gTR_{4n+2}\) are those of \(TG_{8n+4}\) applied by a substitution \((A := \alpha, B := \beta/2, C := \beta/2)\).

(5) is proved by applying spherical cosine law for angles [17, Proposition 2.8] to the following isosceles triangle \(T\): The rhombic tile of \(gTR_{4n+2}\) is subdivided into two copies of \(T\) such that the base edge has length \(\sqrt{2a}\), the non-base inner angle is \(\alpha\) radian, and the two base inner angles are \(\beta/2\) radian.

Recall the construction (Subsection 2.5) of the tiling \(gTR_{4n+2}\). The western hemisphere \(\mathcal{W}\) of \(gTR_{4n+2}\) is that of a spherical monohedral rhombus-faced tiling \(TR_{2n/(2n+1)}\) (see the first and the second images of Figure 4). We set \(xyz\)-coordinate system so that the north pole \(N\) is \((0, 0, 1)\), the part of the unit sphere with positive \(x\)-coordinates is the front hemisphere, and the hemisphere \(\mathcal{W}\) is the part of the unit sphere with negative \(y\)-coordinates.

The vertices of the western hemisphere \(\mathcal{W}\) are: the north pole \(u_0\), the south pole \(u_{2n+1}\), and \(v_i\) \((0 \leq i \leq 2n + 1)\) where the angle from \(u_0\) to each \(v_i\) is \(\pi/2 - (-1)^i(\pi/2 - \alpha)\) and the longitude of each \(v_i\) from the boundary of \(\mathcal{W}\) is \(-i\beta/2\) radian. So, the cartesian coordinates of the vertices on \(\mathcal{W}\) are as in (6).

The edges of the hemisphere \(\mathcal{W}\) are:

\[
\{u_0, v_k\} \quad (k = 0, 2, 4, \ldots, 2n),
\]
(8)

\[
\{u_{2n+1}, v_l\} \quad (l = 1, 3, 5, \ldots, 2n + 1),
\]
(9)

\[
\{v_j, v_{j+1}\} \quad (j = 0, 1, 2, \ldots, 2n).
\]
(10)

The eastern hemisphere \(\mathcal{E}\) of \(gTR_{4n+2}\) is the image of the western hemisphere \(\mathcal{W}\) by the orthogonal transformation \(T_yK\). Here \(K\) is the \(\pi\)-radian rotation operation around the \(z\)-axis, and \(T_y\) is the rotation in \(\alpha\) radian around the \(y\)-axis from the \(z\)-axis toward the \(x\)-axis.

So the vertices of \(gTR_{4n+2}\) on the hemisphere \(\mathcal{E}\) are

\[
u_i := T_yKv_i \quad (0 \leq i \leq 2n + 1).
\]
(11)

Hence the cartesian coordinate of \(u_i\) is (7). By this, \(u_0\) and \(u_{2n+1}\) are indeed the north pole and the south pole, and thus \(T_yKN = v_0\) and \(T_yKS = v_{2n+1}\).

The edges of the eastern hemisphere \(\mathcal{E}\) are the edges (8), (9), and (10) with \(u\) and \(v\) swapped.
This is exactly the graph of Figure 4 (right).

We will prove the angle assignment of \(gTR_{4n+2}\) is as in Figure 4 (right), as follows: If an inner angle of \(gTR_{4n+2}\) corresponds to some inner angle of the western hemisphere of \(TR_{4n/(2n+1)}\), then the type of the inner angle is \(x\) or \(\beta\) as in Figure 4 (right). The two newly arisen inner angles \(\angle v_1v_0u_2n\) and \(\angle u_1u_0v_2n\) of \(gTR_{4n+2}\) near the boundary of \(W\) have the desired values. For example, \(\angle v_1v_0u_2n = \angle v_1v_0u_{2n+1} + \angle u_{2n+1}v_0u_2n = \angle v_1v_0u_{2n+1} + \angle v_2n v_0v_{2n+1} = \angle v_0u_0v_2\) and so on.

We will prove that the Schönflies symbol of \(gTR_{4n+2}\) is \(D_2\), by the procedure [5, p. 55 and Figure 3.10 (p. 56)], as follows:

If \(gTR_{4n+2}\) has a \(k(\geq 2)\)-fold axis \(p\) of rotation, then \(k = 2\). To see it, let \(p\) be a point where \(p\) and \(gTR_{4n+2}\) intersects. If \(p\) is an inner point of a tile of \(gTR_{4n+2}\), then \(k = 2\) because the tile is a rhombus. If \(k > 2\), then \(p\) is not a vertex of the tiling \(gTR_{4n+2}\), because each vertex type of the tiling is of the form \(iA + mB\) for some different inner angles \(A, B\), some positive integer \(m\), and some positive integer \(i < 3\). If \(p\) is on a edge, then \(k = 2\) clearly.

\(gTR_{4n+2}\) has three mutually perpendicular two-fold axes of rotation:

1. The \(y\)-axis.

   The \(\pi\)-radian rotation around it transforms the vertex \(v_i\) to the vertex \(v_{2n+1-i}\) and the vertex \(u_i\) to the vertex \(u_{2n+1-i}\). This rotation preserves the edges of \(gTR_{4n+2}\). Indeed, in the western hemisphere, this rotation transforms the edges (8) to the edges (9) and vice versa, and preserves the edges (10). In the eastern hemisphere, this rotation transforms the edges similarly.

2. \(\ell\) through the midpoint of \(u_0\) and \(v_0\).

   The \(\pi\)-radian rotation around it is a transformation \(T_{\pi/2}KT_{-\pi/2}\).

   The transformation maps the vertex \(u_i = T_{\pi/2}Kv_i\) \((0 \leq i \leq 2n + 1)\) to \((T_{\pi/2}K)^2v_i = v_i\), and vice versa. Clearly, this rotation preserves the edges of \(gTR_{4n+2}\).

3. \(\ell'\) through the midpoint of \(u_0\) and \(v_{2n+1}\).

   The \(\pi\)-radian rotation around it is the composition of the two \(\pi\)-radian rotations around the \(y\)-axis and \(\ell\).

To show that the tiling \(gTR_{4n+2}\) has no mirror plane passing between two of the three two-fold axes, it is sufficient to show that \(v_0u_0v_{2n+1}u_{2n+1}\) is a proper rectangle, because the symmetry of \(gTR_{4n+2}\) fixes the proper rectangle. Here \(v_0u_0v_{2n+1}u_{2n+1}\) is indeed a rectangle, because the vertex \(v_0\) is antipodal to \(v_{2n+1}\) by (6) and (7) and the vertex \(u_0\) is antipodal to \(u_{2n+1}\). The rectangle is not regular, because the edge-length \(a\) of \(gTR_{4n+2}\) is not \(\pi/2\) radian.

The \(xz\)-plane containing the two axes \(\ell\) and \(\ell'\) of rotation is not a mirror plane of \(gTR_{4n+2}\). To see it, first observe that the \(y\)-coordinate of the vertex \(v_0\) of \(gTR_{4n+2}\) is 0. Among the vertices adjacent to \(v_0\), the number of vertices
having positive \( y \)-coordinates is, however, greater than that of vertices having negative \( y \)-coordinates, by (7) and \( \beta = 2/(2n + 1) \). Therefore the Schönflies symbol of \( gTR_{4n+2} \) is \( D_2 \).

\( gTR_{4n+2} \ (n \geq 2) \) is not isohedral, because the number \( 4n + 2 \) of tiles does not divide the order four of the symmetry group \( D_2 \). This completes the proof of Lemma 9.

We finish the proof of Theorem 2 (2), by verifying the data of the following spherical monohedral rhombus-faced tilings given in Table 1:

(1) By Fact 1, the Schönflies symbol of \( TR_{6}^{2/3} \) is \( O_h \), and the symmetry group transitively acts on the tiling \( TR_{6}^{2/3} \).

Consider a spherical monohedral rhombus-faced tiling \( TR_{2n}^{1-1/n} \) with \( n > 3 \). Then, the list of three inner angles around any three-valent vertex of \( TR_{2n}^{1-1/n} \) is \( (\alpha, \alpha, \delta) = (1 - 1/n, 1 - 1/n, 2/n) \), where \( \alpha \neq \delta \) because \( n > 3 \). Hence, we can prove that the Schönflies symbol of \( TR_{2n}^{1-1/n} \) is \( D_{nd} \) and the tiling \( TR_{2n}^{1-1/n} \) is isohedral, as in the proof for the spherical monohedral kite-faced tiling \( TR_{4n}^{1-1/2} \) of Theorem 2 (2).

(2) The Schönflies symbol of the spherical tiling \( F_{24} \) (\( F_{I60} \), resp.) by congruent isosceles triangles is \( O_h \) (\( I_h \), resp), because of [5, p. 50] and the following:

**Lemma 10.** If \( \mathcal{P} \) is the central projection of some Platonic solid to the sphere, then every symmetry operation of \( \mathcal{P} \) is that of \( \text{kis}(\mathcal{P}) \), and \( \text{kis}(\mathcal{P}) \) is isohedral.

**Proof.** Every symmetry operation of \( \mathcal{P} \) is that of \( \text{kis}(\mathcal{P}) \), because it transforms a tile of \( \mathcal{P} \) to a tile of \( \mathcal{P} \), a vertex of \( \mathcal{P} \) to a vertex of \( \mathcal{P} \), and the center of a tile of \( \mathcal{P} \) to the center of a tile of \( \mathcal{P} \).

Suppose a tile \( T \) of \( \mathcal{P} \) is a regular \( p \)-gon. By Fact 4, within the tile \( T \), the \( p \) tiles of \( \text{kis}(\mathcal{P}) \) rotated to each other around a \( p \)-fold axis. Moreover \( T \) is transformed to another tile of \( \mathcal{P} \), by a symmetry operation of \( \mathcal{P} \). Thus, in \( \text{kis}(\mathcal{P}) \), any tile is transformed to any other tile by a symmetry operation of \( \text{kis}(\mathcal{P}) \).

The Schönflies symbol of \( R_{12} \) (\( R_{30} \), resp.) is that of \( F_{24} \) (\( F_{I60} \), resp.), because each symmetry operation of \( F_{24} \) (\( F_{I60} \), resp.) preserves the tiles and the edges of \( F_{24} \) (\( F_{I60} \), resp.).

By the construction (Subsection 2.4) of \( R_{12} \) (\( R_{30} \), resp.) and Fact 4, if \( T, T' \) are tiles of \( R_{12} \) (\( R_{30} \), resp.) sharing the center \( c \) of a tile of \( O_8 \) (\( H_{20} \), resp.), then \( T \) is rotated to \( T' \) around the three-fold (five-fold, resp.) axis \( \rho \) of the rotation such that \( \rho \) passes through the center \( c \). If \( T \) and \( T' \) does not share any vertex, then some symmetry operation of \( O_8 \) (\( H_{20} \), resp.) rotates \( T \) to
another tile $T''$ of $R_{12}$ ($R_{30}$, resp.) such that $T''$ and $T'$ share the center of some tile of $\mathcal{C}_8$ ($H_{20}$, resp.).

By Table 2, the list of inner angles of the isosceles triangular tile $T$ of $F_{24}$ ($F_{60}^I$, resp.) is $(A, B, C) = (2/3, 1/4, 1/4)$ ($(2/3, 1/5, 1/5)$, resp.). As we glue two copies of $T$ at the base edges, the cyclic list of inner angles of the rhombic tile of $R_{12}$ ($R_{30}$, resp.) is $(\alpha, \beta, \alpha)$ is $(2\beta, A, 2B, A)$. So, the vertex types of $R_{12}$ ($R_{30}$, resp.) are as in Table 1, by Fact 2.

4. The anisohedral spherical triangles

We will prove that the tiles of the following spherical monohedral triangular tilings are not anisohedral.

**Lemma 11.** The followings are spherical isohedral triangular tilings:

1. Every instance of a continuously deformable tiling $F_4$, $F_{12}^I$, $F_{24}$, $F_{48}$, $F_{60}$, $F_{60}^I$, and $F_{120}$.

2. Every instance of a continuously deformable tiling $G_{4n}$ ($n \geq 2$), and $G_{4n+2}$ ($n \geq 1$).

3. An instance $H_{20}$ of an infinite series $H_{4n}$ ($n \geq 3$).

4. An instance $I_{24}$ of an infinite series $I_{8n}$ ($n \geq 3$).

**Proof.** (1) Let $\mathcal{T}$ be an instance of $F_4$. $\mathcal{T}$ has two two-fold axes of rotation, each of which is through the midpoint of an edge and that of the opposite edge, in view of the development map of $F_4$ [16, Figure 17]. We can transform any tile $T$ of $\mathcal{T}$ to any other tile $T'$, by $\pi$-rotation around an axis $\rho$ which is through the midpoint of the edge shared by the two tiles $T$ and $T'$. Hence $F_4$ is isohedral.

Next, we prove that $F_{48}$ is isohedral. Indeed, every symmetry operation of $\mathcal{C}_8$ is that of $F_{48}$, as in the proof of Lemma 1.

For each tile $T$ of $\mathcal{C}_8$ and each vertex $A$ of $T$, there is a mirror plane $\sigma$ passing through $A$ and bisecting the edge of $T$ opposite to $A$, because of [5, (ix) of p. 48]. Let $M$ be the midpoint of the edge. The mirror plane $\sigma$ passes through $M$ and the center $c$ of $T$.

Within the tile $T$ of $\mathcal{C}_8$, a pair of right scalene triangular tiles of $F_{48}$ adjacent at the edge $cM$ are reflected to each other by the mirror plane $\sigma$, and another pair of right scalene triangular tiles of $F_{48}$ adjacent at their hypotenuses $cA$ are reflected to each other by another mirror plane $\sigma$. Therefore within the tile $T$ of $\mathcal{C}_8$, the six right scalene triangular tiles of $F_{48}$ are transformed to each other by repeated reflection operations with appropriate mirror planes $\sigma$'s of $\mathcal{C}_8$ where all such $\sigma$'s pass through the center $c$ of $T$. 


Consider that tiles $S$ and $S'$ of $F_{48}$ are in different tiles $T$ and $T'$ of $O_{8}$. Then, to $S$ apply a symmetry operation $\rho$ that maps $T$ to $T'$, and then to the resulting tile of $F_{48}$ apply a symmetry operation mentioned above in order to get $S'$. Hence, $F_{48}$ is isohedral.

The proof of the isohedrality of $F_{120}$ is obtained from the proof of the isohedrality of $F_{48}$, by replacing $F_{48}$ with $F_{120}$, $O_{8}$ with $H_{20}$, and "(ix) of p. 48" with "(v) and (vi) of p. 49."

The isohedrality of $F_{12}$, $F_{24}$, $F_{60}$, and $F_{120}$ is by Lemma 10 and their constructions (cf. $F_{120} = \text{kis}(D_{12})$ by [16, Figure 17 and p. 484]).

(2) An instance $T$ of $G_{4n}$ has $n$ horizontal two-fold axes of rotation and $n$ vertical mirror planes passing between two horizontal two-fold axes, according to [16, Figure 4]. Let $T$ and $T'$ be any tiles of $T$. If $T'$ shares a meridian edge $e$ of $T$, then $T$ is reflected to $T'$ by a vertical mirror plane $\sigma$ of $T$ such that $\sigma$ contains $e$. Moreover, if $T'$ shares a non-meridian edge $e'$ of $T$, then there is a horizontal two-fold axis $\rho$ of rotation of $\sigma$ such that $\rho$ passes through the midpoint of $e'$ and $T$ is rotated to $T'$ around $\rho$.

Hence, any tile of $T$ is transformed to any other tile of $T$ by a symmetry operation of $T$. Thus, $T$ is isohedral.

$G_{4n+2}$ ($n \geq 1$) is isohedral by the vertical $(2n + 1)$-fold axis of rotation and the horizontal mirror plane, according to [16, Figure 4].

(3) An instance $H_{20}$ is the central projection of the regular icosahedron to the sphere, and thus the Schönflies symbol of $H_{20}$ is $I_{h}$, and $H_{20}$ is isohedral.

(4) This is due to Proposition 2 (1) and Lemma 10.

The other spherical triangular tilings are not isohedral.

**Lemma 12.** Spherical monohedral triangular tilings $H_{4n}$ ($5 \neq n \geq 3$), $I_{8n}$ ($n \geq 4$), $TH_{8m+4}$ ($m \geq 3$), and $TI_{16m+8}$ ($m \geq 2$) are non-isohedral.

**Proof.** For $T = H_{4n}$ ($5 \neq n \geq 3$) or $I_{8n}$ ($n \geq 4$), the Schönflies symbol of $T$ is $D_{nd}$. It is due to Proposition 2, because every symmetry operation of $TR_{2n}^{1-1/n}$ preserves the diagonal segments of the tile and the Schönflies symbol of $TR_{2n}^{1-1/n}$ is $D_{nd}$ by Theorem 2.

$I_{8n}$ ($n \geq 4$) is not isohedral, since the order of the symmetry group $D_{nd}$ of $I_{8n}$ is $4n$ according to [5, p. 44] but the number $8n$ of the tiles of $I_{8n}$ does not divide $4n$.

$H_{4n}$ ($5 \neq n \geq 3$) is not isohedral, because no symmetry operation transforms a tile around the pole to a tile on the equator (See the first line of Figure 7).

We can prove that the Schönflies symbol of $TH_{8m+4}$ ($m \geq 3$) ($TI_{16m+8}$ ($m \geq 2$), resp.) is $D_{2}$, in a similar manner as we proved that the Schönflies
symbol of \( gTR_{4n+2} \) is \( D_2 \) in Lemma 9, in view of the construction (Proposition 2 (2)) of \( TH_{8m+4} \) (\( TI_{16m+8} \), resp.).

\[ TH_{8m+4} \ (m \geq 3) \ (TI_{16m+8} \ (m \geq 2), \text{resp.}) \] is not isohedral, because the number \( 8m + 4 \ (m \geq 3) \ (16m + 8 \ (m \geq 2), \text{resp.}) \) of tiles does not divide the order 4 of the symmetry group \( D_2 \).

Although \( H_{12} \) is not isohedral, the tile of \( H_{12} \) is congruent to the tile of the tiling \( G_{12} \) with \( A = 2/3, B = C = 1/3. \ G_{12} \) is isohedral by Lemma 11 (2).

The other spherical monohedral, non-isohedral triangular tilings listed in the previous lemma are spherical tilings by anisohedral triangles, owing to the following lemma:

**Lemma 13.** (1) If the tile of \( H_{4n} \ (5 \neq n \geq 4) \) is the tile of another spherical monohedral triangular tiling \( \mathcal{T} \), then \( \mathcal{T} \) is \( TH_{8m+4} \) for some \( m \geq 3 \) with \( 4n = 8m + 4 \).

(2) If the tile of \( TH_{8m+4} \ (m \geq 3) \) is the tile of another spherical monohedral triangular tiling \( \mathcal{T} \), then \( \mathcal{T} \) is \( H_{8m+4} \).

(3) If the tile of \( I_{8n} \ (n \geq 4) \) is the tile of another spherical monohedral triangular tiling \( \mathcal{T} \), then \( \mathcal{T} \) is \( TI_{16m+8} \) for some \( m \geq 2 \) with \( 8n = 16m + 8 \).

(4) If the tile of \( TI_{16m+8} \ (m \geq 2) \) is the tile of another spherical monohedral triangular tiling \( \mathcal{T} \), then \( \mathcal{T} \) is \( I_{16m+8} \).

**Proof.** (1) The number of tile of \( \mathcal{T} \) is \( 4n \ (5 \neq n \geq 4) \) and the tile of \( H_{4n} \) is an isosceles triangle. So, if \( \mathcal{T} \) is sporadic, then \( n \geq 6 \) and \( \mathcal{T} \) is \( F_{24}, F_{60}^I \) or \( F_{60}^{II} \). However, the non-base inner angle of \( \mathcal{T} \) is greater than the non-base inner angle \( 2/n \) of \( H_{4n} \) for each case. Thus \( \mathcal{T} \) is a non-sporadic spherical tiling by congruent isosceles triangles. Hence, \( \mathcal{T} \) is \( G_{4n}, TG_{4n}, \text{or } TH_{8m+4} \) for some \( m \geq 3 \) such that \( 8m + 4 = 4n \). By [16, Table] (included as Table 2), the tile of \( G_{4n} \) and that of \( TG_{4n} \) have an inner angle \( 1/n \) but the tile of \( H_{4n} \) does not, as \( 5 \neq n \geq 4 \). Thus \( \mathcal{T} \) is \( TH_{8m+4} \), as desired.

(3) The number of tile of \( \mathcal{T} \) is \( 8n \geq 32 \) and the tile of \( I_{8n} \) is a right scalene triangle. So, if \( \mathcal{T} \) is sporadic, then (1) \( n = 6 \) and \( \mathcal{T} \) is \( F_{48} \) or \( TF_{48} \), or (2) \( n = 15 \) and \( \mathcal{T} = F_{120} \). Then the smallest inner angle of \( \mathcal{T} \) is greater than the smallest inner angle \( 1/n \) of \( I_{8n} \). Thus \( \mathcal{T} \) is a non-sporadic spherical tiling by congruent right scalene triangles. So \( \mathcal{T} \) is \( G_{8n}, \text{or } TI_{8n} \). By [16, Table] (included as Table 2), an inner angle \( 1/(2n) \) of the tile of \( G_{8n} \) is smaller than the smallest angle \( 1/n \) of the tile of \( I_{8n} \) for each case. So the tile of \( \mathcal{T} \) is not congruent to the tile of \( G_{8n} \). Thus \( \mathcal{T} = TI_{8n} \), as desired.

Assertion (2) (assertion (4), resp.) follows from assertion (1) (assertion (3), resp.), because the tile of \( TH_{8m+4} \ (m \geq 3) \ (TI_{16m+8} \ (m \geq 2), \text{resp.}) \) is also the tile of \( H_{8m+4} \ (I_{16m+8}, \text{resp.}) \).
Lemma 14. The spherical tilings by congruent anisohedral triangles are exactly

\[ H_{4n} (n = 4, 6) \quad TH_{8m+4} (m = 3, 4, 5), \quad I_{8n} (n = 4, 5) \quad TI_{16m+8} (m = 2, 3, 4), \quad \text{and} \quad TI_{16m+8} (m = 2, 3, 4). \]

Proof. \[ H_{4n} (5 \neq n \geq 4), \quad TH_{8m+4} (m \geq 3), \quad I_{8n} (n \geq 4), \quad \text{and} \quad TI_{16m+8} (m \geq 2). \]

We will prove that for any other \( T \) listed in Ueno-Agaoka’s complete classification ([16, Table], included as Table 2) of spherical monohedral triangular tilings but not listed in Lemma 11, the tile \( T \) of \( T \) is congruent to some spherical isohedral triangular tiling. We will argue such \( T \) from top to bottom of Table 2. If \( T \) is \( F_{12}^{\text{II}} \), or \( F_{12}^{\text{III}} \), then the tile \( T \) of \( T \) admits \( F_{12}^{I} \), because the list of three inner angles of the tile \( T \) is the list of three inner angles of the tile \( F_{12}^{I} \). As \( F_{12}^{I} \) is isohedral by Lemma 11 (1), the tile of \( T \) is not anisohedral. The tile of \( TF_{48} \) admits \( F_{48} \), because the list of three inner angles of \( T \) is the list of three inner angles of the tile of \( F_{48} \). \( F_{48} \) is isohedral by Lemma 11 (1). Hence, the tile of \( TF_{48} \) is not anisohedral. If \( T \) is \( TG_{8n} \)
(n ≥ 2), \( TG_{8n+4} \) (n ≥ 1), \( MTG^I_{8n+4} \) (n ≥ 1), \( MTG^II_{8n+4} \) (n ≥ 2), or \( H_{12} \), then the tile \( T \) of \( \mathcal{T} \) admits an instance of \( G_m \) where \( m \) is the number of tiles of \( \mathcal{T} \). \( \mathcal{T} \) is isohedral by Lemma 11 (2). Hence, the tile of \( \mathcal{T} \) is not anisohedral.

Because Ueno-Agaoka’s complete classification of spherical monohedral triangular tilings describes the inner angles of the tiles, Theorem 1 follows from the previous Lemma (see Figure 7).

**Theorem 3.** No spherical (kite/dart/rhombus) is anisohedral.

**Proof.** By Theorem 2 (2) and Table 1, the tiles of the only spherical monohedral, non-isohedral (kite/dart/rhombus)-faced tilings \( gK_{24} \) and \( gTR_{4n+2} \) (n ≥ 2) are those of spherical isohedral rhombus-faced tilings \( K_{24} \) and \( TR_{4n+2}^{2n/(2n+1)} \).

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References

Anisohedral spherical triangles, kites, darts and rhombi


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