Scratching-sound generator based on the logistic map*

Yasushi MIKI

Abstract

A mathematical model for generating annoying or chilling sounds is presented. Such sounds are generated by frictional motion and are generally considered to have chaotic properties. The proposed model is based on the logistic map and is modified to have the stick-slip property of a frictional vibration. A joystick is used to control the gain parameter, which determines the chaotic behavior of the logistic map, and the velocity parameter, which determines the time interval of transition. The obtained sound is similar to that generated by scratching a chalkboard or glass plate with fingernails.

Keywords: Annoying; Chilling; Scratching sound; Logistic map; Stick-slip; Frictional vibration

1. Introduction

Recent research on annoying sounds generated by, for example, scratching a chalkboard with fingernails have been presented from the viewpoint of psychoacoustics.[1,2] We have previously examined those sounds from a physical standpoint.[3] In the present paper, however, we propose a mathematical model for generating a scratching sound. Such a sound is generated from the stick-slip motion of a frictional system, and it is shown herein that the mechanism thereof is strongly related to the logistic map. Since the logistic map provides only a periodic slip action, an additional property has to be introduced to produce the stick-slip action, which is similar to the bowing action of string instruments.

2. Frictional motion and the logistic map

The particle motion in one dimension under the influence of periodic impulses has been described by Milonni, Shih, and Ackerhalt[4] as follows.

2.1. Non-dissipative system

Let the mass of the particle be m and the force be

$$F = A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n), \tag{1}$$

where $A(x)$ is an arbitrary function of position x and $\delta(t)$ is the Dirac delta function. Then the Hamilton equations of motion for the position and momentum are

$$\dot{x} = \frac{p}{m}, \tag{2a}$$

$$\dot{p} = A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n). \tag{2b}$$

Integrating these equations from $t=nT-\varepsilon$ to $(n+1)\,T-\varepsilon$, where ε is an infinitesimally short time, we have

$$x_{n+1} = x_n + \frac{T}{m}\left[p_n + TA(x_n)\right], \tag{3a}$$

$$p_{n+1} = p_n + TA(x_n), \tag{3b}$$

where x_n and p_n are the values of x and p just before the n-th kick. Details are described in the Appendix. This is a discrete mapping of a periodically kicked system.

2.2. Dissipative system

Next, we consider the system with equations of motion

$$\dot{x} = \frac{p}{m}, \tag{4a}$$

$$\dot{p} = -\beta p + A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n), \tag{4b}$$

which differs from Eq. (2) by the presence of frictional term $-\beta p$ in the momentum equation. Integrating these equations, we have

$$x_{n+1} = x_n + \frac{1}{m\beta}\left(1-\mathrm{e}^{-\beta T}\right)\left[p_n + TA(x_n)\right], \tag{5a}$$

$$p_{n+1} = \mathrm{e}^{-\beta T}\left[p_n + TA(x_n)\right]. \tag{5b}$$

Details are described in the Appendix.

Equation (5) reduces to the mapping (3) in the limit $\beta \to 0$ of no friction. If we assume

$$A(x) = \frac{m\beta}{T}\left(4\lambda x(1-x) - x\right) \tag{6}$$

and consider the limit $\beta \to \infty$ of strong damping, we have $p_{n+1} \to 0$ and

$$x_{n+1} = 4\lambda x_n\left(1-x_n\right). \tag{7}$$

This equation is known as the logistic map. Thus, the logistic map has been derived from the strong-damping limit of a periodically kicked system.

3. Scratching-sound generator

3.1. Chaotic behavior of the logistic map

The logistic map is often used to show chaotic behavior, and it is found to be strongly related to frictional vibration. The sound generated by the logistic map with the parameter λ in Eq. (7) varied from 0.75 to 1 is presented in the *Mathematica* document.[5] A slight modification of the program shown below leads to a sound similar to a scratching sound (Fig. 6 (a) in Section 4):

```
logistic[n_Integer]: = Module[{f,t,x},
  f=Compile[{x,t},Evaluate[(3.6+t/n/5)*x*(1-x)]];
FoldList[f,0.223,Range[n]]]
ListPlay[logistic[32000],SampleRate->16000]
```

* 原稿受付 平成27年11月14日

One of the major modifications is that the sampling frequency is increased to 16 kHz. An annoying or chilling impression seems to occur when the logistic system enters or leaves chaos.

3.2. Representation of stick-slip motion

The scratching model described above is a periodic slip model. Frictional motions in general are described as stick-slip phenomena. One of the most beautiful sounds might be considered to be that of a violin or other bowed string-generated sound. On the other hand, the sound produced by scratching a chalkboard with fingernails might be considered the worst. In both cases, however, the mechanism for producing sounds is the same, i.e., the stick-slip phenomenon of a frictional system. The stick-slip phenomenon is a repetition of two stages: (a) a stick stage and (b) a slip stage.

(a) The stick stage represents the stick motion moving at a constant speed along with, for example, the bow and continues until a slip occurs.

(b) The slip stage represents the slip motion whose movement is immediate and stops due to strong damping.

The timing of the transition in the logistic model is assumed to be periodic. To import the stick-slip property into the model, we introduce the following rules:

(a) If $x_{n+1} < x_n$ in Eq. (7), the value of x within (x_{n+1}, x_n) decreases linearly at a constant speed. This is the stick motion and corresponds to the down stroke of the bow.

(b) If $x_{n+1} \geq x_n$, transition occurs immediately. This is the slip motion.

Thus we obtain the modified logistic map representing the stick-slip phenomenon.

3.3. The scratching-sound generator

The flow chart of the new logistic model discussed above is shown in Fig. 1. The left branch describes the stick stage and the right branch describes the slip stage. The velocity parameter Δx gives the bowing speed of the down stroke.

Figure 2 shows the behavior of the basic logistic model with $\lambda = 0.85$. A stable and periodic oscillation is observed. Figure 3 shows the behavior of the modified model with the same λ value. A sawtooth waveform is produced with the period determined by the velocity parameter Δx.

If λ is changed to 0.95, the system enters the chaos state,

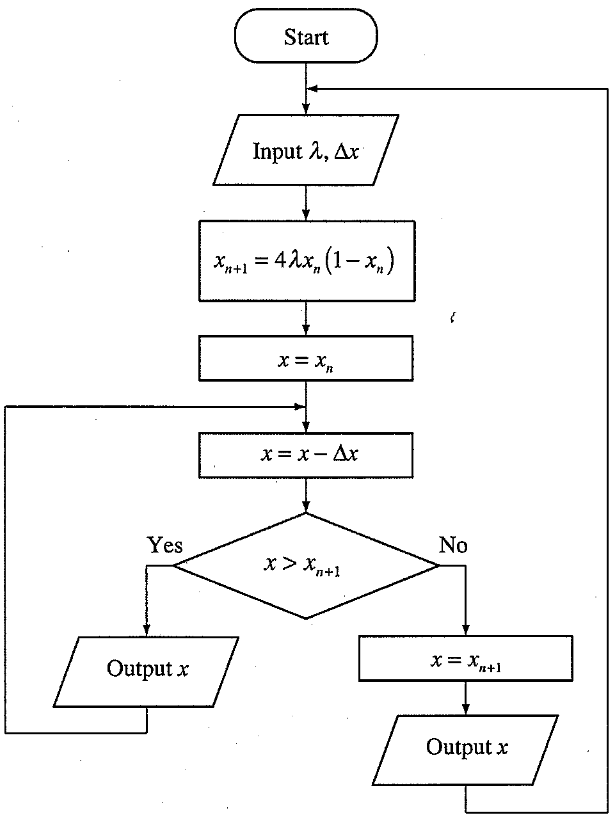


Fig. 1. Flowchart of the scratching-sound generator. λ is the gain parameter, Δx the velocity parameter, and x the output data.

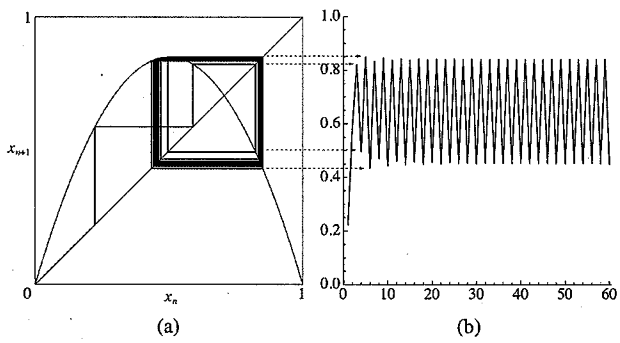


Fig. 2. (a) Basic logistic map and (b) the generated waveform. $\lambda = 0.85$, $x_0 = 0.223$.

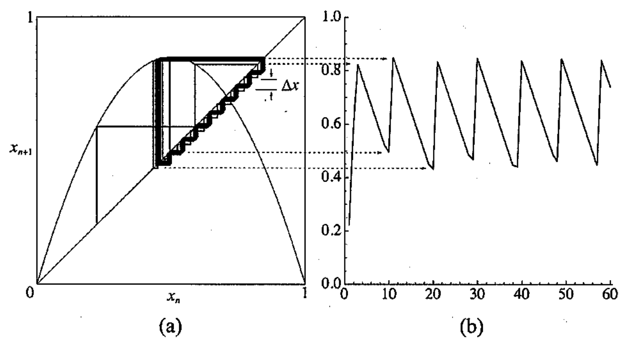
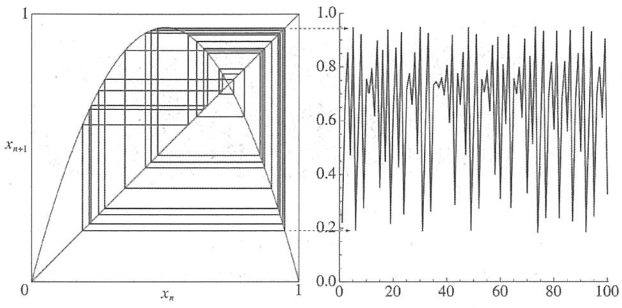
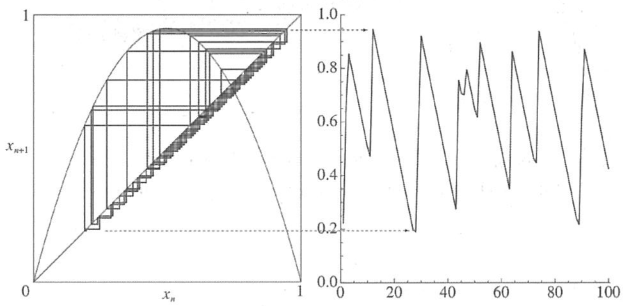


Fig. 3. (a) Modified logistic map representing the stick-slip phenomenon and (b) the generated waveform. $\lambda = 0.85$, $x_0 = 0.223$, $\Delta x = 0.05$, where Δx corresponds to the bowing speed.



(a) (b)

Fig. 4. Basic logistic model in chaos. $\lambda = 0.95$.



(a) (b)

Fig. 5. Modified logistic model in chaos. $\lambda = 0.95$.

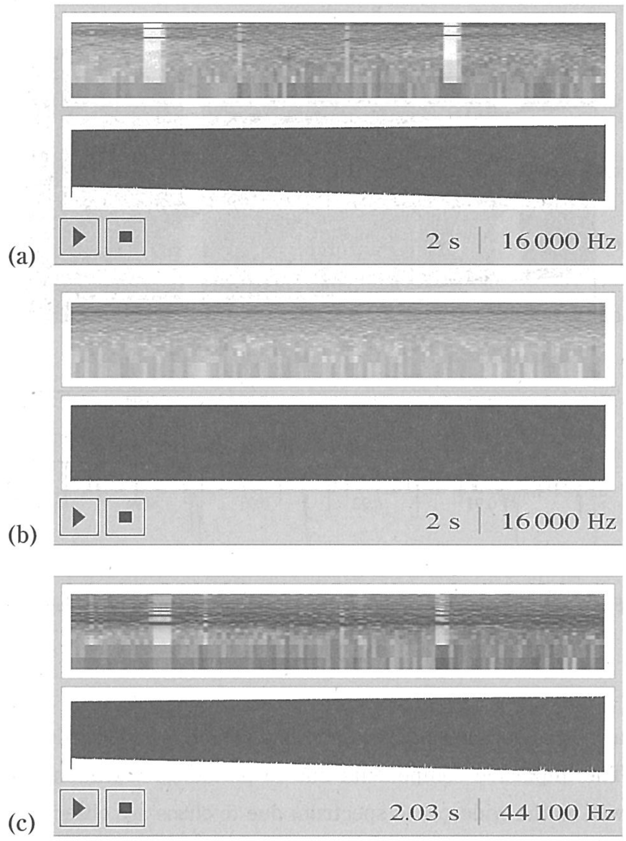


Fig. 6. (a) and (b) Sound clips generated by the original logistic model at a sampling rate of 16 kHz with λ varied from 0.9 to 0.95, and $\lambda = 0.9$, respectively. (c) Sound clip generated by the proposed stick–slip model with λ varied from 0.9 to 0.95 at a sampling rate of 44.1 kHz and with the additional parameter Δx set to 0.05.

as shown in Figs. 4 and 5. The scratching-sound generator is intended to allow free control of the value of λ by use of a joystick, thus allowing control of the system between the stable state and the chaos state. The joystick system also allows velocity and pitch fluctuation control.

4. Discussions

Scratching sounds often produce a strongly uncomfortable sensation. But what causes this feeling remains to be understood. This paper does not intend to answer this question, but instead presents a model to produce such sounds. As described above, scratching sounds are strongly related to chaos, but chaos in a steady state does not necessarily give a chilling impression. Entering or leaving chaos might be the main factor causing such impressions.

Let us examine the sounds produced by the method described above. Following sound clips are embedded in the pdf version of this article [6] and also available in the *Mathematica* document [7] by the author.

Figure 6 (a) shows the sound clip generated by the logistic map with λ varied from 0.9 to 0.95 at a sampling rate of 16 kHz. In Fig. 7, the bifurcation diagram and the corresponding Lyapunov exponents [8] are shown. If the Lyapunov exponent is positive, the system is chaotic; and if it is negative, the system becomes stable and converges to a periodic state. It is found that stable states appear intermittently during the chaos state.

Figure 6 (b) shows the sound clip of a chaotic sound in steady state with $\lambda = 0.9$. It sounds like a stationary random noise, and the chilling impression is less than (a). This result implies that the chaos itself does not necessarily cause chills, but the transition from and into chaos seems to be the major factor of the chilling sensation.

Figure 6 (c) shows the sound clip generated by the proposed stick–slip model with λ varied from 0.9 to 0.95 at a sampling rate of 44.1 kHz and with the additional parameter Δx set to 0.05. The sound impression does not differ much from that of Fig. 6 (a), but this model allows the control of pitch frequency.

Let us consider the sound from a viewpoint of frequency spectrum. Figures 8 and 9 show the spectrogram and the

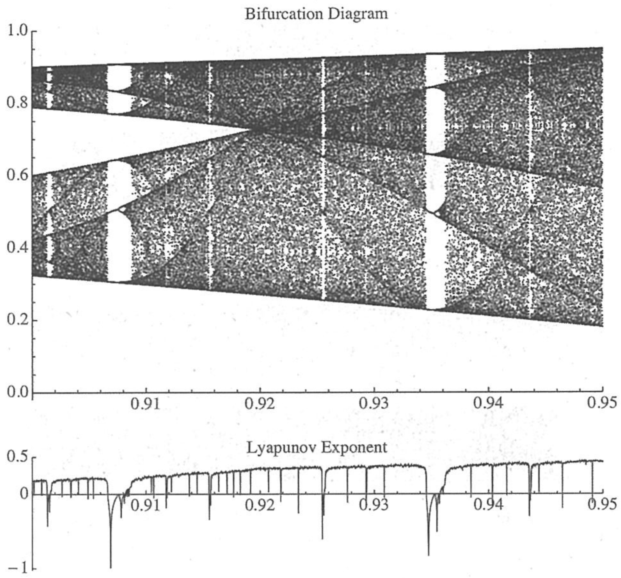


Fig. 7. Bifurcation diagram and the Lyapunov exponent of the logistic map for λ varied from 0.9 to 0.95.

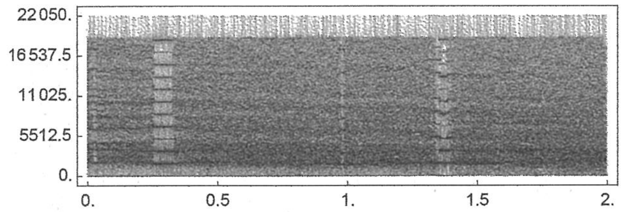


Fig. 8. Spectrogram of the sound clip shown in Fig. 6 (c). Holizontal axis: time in seconds. Vertical axis: frequency in Hz.

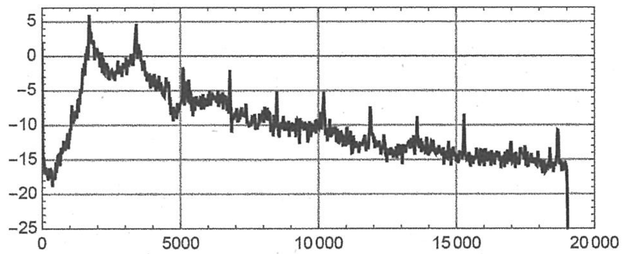


Fig. 9. Average spectram of the sound clip shown in Fig. 6 (c). Holizontal axis: frequency in Hz. Vertical axis: amplitude in dB.

average spectrum of the sound clip (Fig. 6 (c)) respectively. The impulsive components due to periodical vibrations as well as the widespread spectrum due to chaos are observed.

5. Conclusion

It was shown that the logistic map is derived from a periodically kicked frictional motion, and that a mathematical model based on the logistic map produces such sounds as are generated by scratching a chalkboard or glass plate with fingernails. This model exhibits chaotic properties, and chilling impressions are found to be strongly related to chaotic behavior, espe cially the behavior of entering chaos and/or leaving chaos. A scratching-sound generator was described that uses a joystick to control the gain parameter of the logistic map as well as the velocity parameter. The latter parameter determines the time interval spent in the stick state. However, why scratching sounds produce annoying or chilling impressions still remains to be understood.

Acknowledgement

The author would like to express his gratitude to the late Dr. Yasushi Ishii, who prompted him to start this research 40 years ago and had been encouraging him since.

References

[1] Trevor J. Cox, Scraping sounds and disgusting noises. Appl Acoust 2008, 69, 1195-1204.

[2] D. Lynn Halpern, R. Blake and J. Hillenbrand, Psychoacoustics of a chilling sound. Perception & Psychophysics 1986, 39 (2), 77-80.

[3] H. Murakoshi and Y. Miki, Analysis of uncomfortable sound (in Japanese), Autumn meeting of Acoust. Soc. Jpn., 1993, 699-700.

[4] Peter W. Milonni, Mei-Li Shih and Jay R. Ackerhalt, Chaos in laser-matter interactions. World Scientific Publishing 1987.

[5] Walfram Research, Logistic map. http://reference.wolfram.com/legacy/v5_2/Demos/SoundGallery/LogisticMap.html (Accessed 3 March 2015)

[6] Y. Miki, Scratching-sound generator based on the logistic map.
https://www.takushoku-u.ac.jp/laboratory/files/sci_engnrng_16-1.pdf

[7] Y. Miki, Using the logistic map to generate scratching sounds.
http://www.mathematica-journal.com/2013/05/using-the-logistic-map-to-generate-scratching-sounds/ (Accessed 6 Sept. 2015)

[8] Erik Mahieu, Lyapunov exponents for the logistic map. http://demonstrations.wolfram.com/LyapunovExponentsForTheLogisticMap/(Accessed 3 March 2015)

Appendix

Derivation of the discrete mapping

(1) Non-dissipative system:

Integrating the equation

$$\dot{p} = A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n) \tag{a1}$$

from $t=nT-\varepsilon$ to $(n+1)T-\varepsilon$, where ε is an infinitesimally short time, we have

$$\begin{aligned}[p]_{nT-\varepsilon}^{(n+1)T-\varepsilon} &= A(x)\int_{nT-\varepsilon}^{(n+1)T-\varepsilon}\delta(t/T-n)\,dt \\ &= TA(x)\end{aligned} \tag{a2}$$

$$\therefore \quad p_{n+1} = p_n + TA(x_n). \tag{a3}$$

Similarly, from the equation

$$\dot{x} = \frac{p}{m}, \tag{a4}$$

$$\begin{aligned}[x]_{nT-\varepsilon}^{(n+1)T-\varepsilon} &= \frac{1}{m}\int_{nT-\varepsilon}^{(n+1)T-\varepsilon} p\,dt \\ &= \frac{1}{m}[t\,p]_{nT-\varepsilon}^{(n+1)T-\varepsilon} - \frac{1}{m}\int_{nT-\varepsilon}^{(n+1)T-\varepsilon} t\,\dot{p}\,dt \\ &= \frac{1}{m}\left[(n+1)T\,p_{n+1} - nT\,p_n\right] - \frac{A(x)}{m}\int_{nT-\varepsilon}^{(n+1)T-\varepsilon} t\,\delta(t/T-n)\,dt \\ &= \frac{1}{m}\left[(n+1)T\left(p_n + TA(x_n)\right) - nT\,p_n\right] - \frac{A(x)}{m}nT\cdot T \\ &= \frac{T}{m}\left[p_n + (n+1)TA(x_n) - nTA(x_n)\right] \\ &= \frac{T}{m}\left[p_n + TA(x_n)\right]\end{aligned} \tag{a5}$$

$$\therefore \quad x_{n+1} = x_n + \frac{T}{m}\left[p_n + TA(x_n)\right]. \tag{a6}$$

(2) Dissipative system:

The differential equation for the momentum

$$\dot{p} = -\beta p + A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n) \tag{a7}$$

can be solved as

$$p = e^{-\beta t}\left[\int e^{\beta t}A(x)\sum_{n=-\infty}^{\infty}\delta(t/T-n)dt + C\right], \tag{a8}$$

where C is an arbitrary constant. Then

$$\left[e^{\beta t}p\right]_{nT-\varepsilon}^{(n+1)T-\varepsilon} = \int_{nT-\varepsilon}^{(n+1)T-\varepsilon} e^{\beta t}A(x)\delta(t/T-n)dt, \tag{a9}$$

$$e^{\beta(n+1)T}p_{n+1} - e^{\beta nT}p_n = e^{\beta nT}TA(x_n), \tag{a10}$$

$$e^{\beta T}p_{n+1} - p_n = TA(x_n). \tag{a11}$$

$$\therefore \quad p_{n+1} = e^{-\beta T}\left[p_n + TA(x_n)\right]. \tag{a12}$$

Similarly,

$$\begin{aligned}[x]_{nT-\varepsilon}^{(n+1)T-\varepsilon} &= \frac{1}{m}\int_{nT-\varepsilon}^{(n+1)T-\varepsilon} p\,dt \\ &= \frac{1}{m\beta}\int_{nT-\varepsilon}^{(n+1)T-\varepsilon}\left[-\dot{p} + A(x)\delta(t/T-n)\right]dt \\ &= \frac{1}{m\beta}\left[[-p]_{nT-\varepsilon}^{(n+1)T-\varepsilon} + TA(x_n)\right] \\ &= \frac{1}{m\beta}\left[-p_{n+1} + p_n + TA(x_n)\right] \\ &= \frac{1}{m\beta}\left[-e^{-\beta T}\left[p_n + TA(x_n)\right] + p_n + TA(x_n)\right] \\ &= \frac{1}{m\beta}\left(1-e^{-\beta T}\right)\left[p_n + TA(x_n)\right]\end{aligned} \tag{a13}$$

$$\therefore \quad x_{n+1} = x_n + \frac{1}{m\beta}\left(1-e^{-\beta T}\right)\left[p_n + TA(x_n)\right]. \tag{a14}$$