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Cosmetic surgery and the $SL(2,\mathbb{C})$ Casson invariant for two-bridge knots

Dedicated to Professor Makoto Sakuma on the occasion of his 60th birthday

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Abstract. We consider the cosmetic surgery problem for two-bridge knots in the 3-sphere. We first verify by using previously known results that all the two-bridge knots of at most 9 crossings admit no purely cosmetic surgery pairs except for the knot 9_{27}. Then we show that any two-bridge knot corresponding to the continued fraction $[0,2x,2,-2x,2x,2,-2x]$ for a positive integer x admits no cosmetic surgery pairs yielding homology 3-spheres, where 9_{27} appears when $x=1$. Our advantage to prove this is using the $SL(2,\mathbb{C})$ Casson invariant.

1. Introduction

Dehn surgery can be regarded as an operation to make a 'new' 3-manifold from a given one. Of course, the trivial Dehn surgery leaves the manifold unchanged, but 'most' non-trivial ones would change the topological type. In fact, Gordon and Luecke showed as the famous result in [10] that any non-trivial Dehn surgery on a non-trivial knot in the 3-sphere S^3 never yields S^3 again.

As a natural generalization, the following conjecture was raised.

Cosmetic Surgery Conjecture ([14, Problem 1.81(A)]): Two surgeries on inequivalent slopes are never purely cosmetic.

Here we say that two slopes are *equivalent* if there exists a homeomorphism of the exterior of a knot K taking one slope to the other, and two surgeries on K along slopes r_1 and r_2 are *purely cosmetic* if there exists an orientation preserving homeomorphism between the pair of the surgered manifolds.

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Remark 1. The Cosmetic Surgery Conjecture for "chirally cosmetic" case is not true: there exist counter-examples given by Mathieu [16, 17]. In fact, for example, $(18k+9)/(3k+1)$- and $(18k+9)/(3k+2)$-surgeries on the right-hand trefoil knot in S^3 yield orientation-reversingly homeomorphic pairs of 3-manifolds for any non-negative integer k, and it can be shown that such pairs of slopes are inequivalent. That is to say, the trefoil knot admits chirally cosmetic surgery pairs along inequivalent slopes.

In this paper, we consider cosmetic surgeries on a well-known class of knots in S^3, the *two-bridge knots*. First, by using known results, we have the following in Section 2.

Proposition 1. *All the two-bridge knots of at most* 9 *crossings admit no purely cosmetic surgery pairs except for the knot* 9_{27} *in Rolfsen's knot table.*

Here the knot 9_{27} is a two-bridge knot illustrated in Figure 1, which corresponds to the following continued fraction expansion.

$$\frac{18}{49} = [0,2,2,-2,2,2,-2] = 0 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{-2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{-2}}}}}}$$

We refer the knot by $S(49,18)$. Our notations basically follow those in [3], but we use $S(\alpha,\beta)$ instead of $K(\alpha,\beta)$ in [3].

In view of this, let us focus on the knot 9_{27}. Previously, for the same reason, the first author considered this knot in [13], and it was shown that $10/3$- and $-10/3$-surgeries on 9_{27} give distinct manifolds by using non-orientable surfaces in surgered manifolds.

In this paper, for a family of knots including the knot 9_{27}, we have the following.

Theorem 1. *Let* K_x *be the two-bridge knot described by* $S((8x^2-1)^2, 32x^3-8x^2-8x+2)$ *with the continued fraction expansion* $[0,2x,2,-2x,2x,2,-2x]$ *for a positive integer* x. *Then* K_x *admits no purely cosmetic surgery pairs*

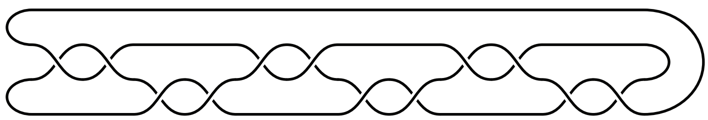

Fig. 1. The knot $9_{27} = S(49,18)$ with the continued fraction expansion $[0,2,2,-2,2,2,-2]$ for $18/49$

yielding homology 3-*spheres, i.e., any* $\frac{1}{n}$- *and* $\frac{1}{m}$-*surgeries on* K_x *are not purely cosmetic for* $m \neq n$. *In other words, all the homology* 3-*spheres obtained by Dehn surgeries on* K_x *are mutually distinct.*

Remark 2. This cannot be achieved by using known invariants; the (original) Casson invariant and the τ-invariant defined by Ozsváth-Szabó in [20], and the correction term in Heegaard Floer homology. See Section 5 for details.

Our advantage in this paper is to use the $SL(2,\mathbb{C})$ version of the Casson invariant. Very roughly speaking, for a closed orientable 3-manifold Σ, the $SL(2,\mathbb{C})$ Casson invariant $\lambda_{SL(2,\mathbb{C})}(\Sigma)$ is defined by counting the (signed) equivalence classes of representations of the fundamental group $\pi_1(\Sigma)$ in $SL(2,\mathbb{C})$. Based on the method to enumerate the boundary slopes for two-bridge knots developed in [18], we give calculations of the $SL(2,\mathbb{C})$ Casson invariant for the knots K_x's. The calculations will be given in Section 4. Before that the formulae and the method used in the calculations will be explained in Section 3.

Remark 3. It is known that the $SL(2,\mathbb{C})$ Casson invariant cannot distinguish the mirror images. In other words, the orientation-reversingly homeomorphic pair of 3-manifolds have the same $SL(2,\mathbb{C})$ Casson invariant. It implies that if a pair of 3-manifolds have different values of the $SL(2,\mathbb{C})$ Casson invariant, then they are not homeomorphic. Thus, as in the statement, we can say that all the homology 3-spheres obtained by Dehn surgeries on K_x are mutually distinct.

Practically our method can be applied further. However it seems not enough to prove that all the K_x's have no purely cosmetic surgery pairs. See Remark 4 in detail.

Here we recall basic definitions and terminology about Dehn surgery.

A *Dehn surgery* is the following operation for a given knot K in a 3-manifold M. Take the exterior $E(K)$ of K, and then, glue a solid torus back to $E(K)$. On the peripheral torus $\partial E(K)$ of K in M, let γ be the slope represented by the curve identified with the meridian of the attached solid torus via the surgery. Then, by $K(\gamma)$, we denote the manifold which is obtained by the Dehn surgery on K along γ, and call it the 3-manifold obtained by Dehn surgery on K along γ. In particular, the Dehn surgery on K along the meridional slope is called the *trivial* Dehn surgery.

When K is a knot in S^3, by using the standard meridian-longitude system, slopes on the peripheral torus are parametrized by rational numbers with $1/0$ (corresponding to the meridian). Thus, when a slope γ corresponds to a rational number r, Dehn surgery along γ is said to be *r-Dehn surgery*, or simply *r-surgery*, and we use $K(r)$ in stead of $K(\gamma)$.

2. Two-bridge knots

For two-bridge knots, see [5] as a standard reference. Our notations basically follows from those in [3].

To show Proposition 1, we use the following two known results.

One ingredient is the Casson invariant of 3-manifolds introduced by Casson. By using the Casson invariant, Boyer and Lines in [4] proved that a knot K in S^3 satisfying $\Delta_K''(1) \neq 0$ has no cosmetic surgeries. Here $\Delta_K(t)$ denotes the (symmetrized) Alexander polynomial for K. That is, $\Delta_K(t)$ satisfies that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

The other one is the following excellent result recently obtained by Ni and Wu in [19]. Suppose that K is a non-trivial knot in S^3 and $r_1, r_2 \in \mathbb{Q} \cup \{1/0\}$ are two distinct slopes such that the surgered manifolds $K(r_1)$, $K(r_2)$ are orientation-preservingly homeomorphic. Then r_1, r_2 satisfy that (a) $r_1 = -r_2$, (b) $q^2 \equiv -1 \mod p$ for $r_1 = p/q$, (c) $\tau(K) = 0$, where τ is the invariant defined by Ozsváth-Szabó in [20]. This result is obtained by using Heegaard Floer homology. We remark that, for alternating knots, $\tau(K) = -\sigma(K)$ holds [20, Theorem 1.4], where $\sigma(K)$ denotes the signature of K.

Now Proposition 1 follows from Table 1. To fill the table, we use the values given in *Knotinfo* [6]. Also we can use the fact that the half of $\Delta_K''(1)$ is equal to the second coefficient of the Conway polynomial. This well-known fact is due to Casson, and, for details, see [1] and [12, Section 1] for example.

Table 1. Two-bridge knots of at most 9 crossings with trivial τ-invariant

Name	$S(\alpha, \beta)$	Alexander Polynomial	$\Delta_K''(1)$
4_1	$S(5,2)$	$-t^{-1} + 3 - t$	-2
6_1	$S(9,7)$	$-2t^{-1} + 5 - 2t$	-4
6_3	$S(13,5)$	$t^{-2} - 3t^{-1} + 5 - 3t + t^2$	2
7_7	$S(21,8)$	$t^{-2} - 5t^{-1} + 9 - 5t + t^2$	-2
8_1	$S(13,11)$	$-3t^{-1} + 7 - 3t$	-6
8_3	$S(17,4)$	$-4t^{-1} + 9 - 4t$	-8
8_8	$S(25,9)$	$2t^{-2} - 6t^{-1} + 9 - 6t + 2t^2$	4
8_9	$S(25,7)$	$-t^{-3} + 3t^{-2} - 5t^{-1} + 7 - 5t + 3t^2 - t^3$	-4
8_{12}	$S(29,12)$	$t^{-2} - 7t^{-1} + 13 - 7t + t^2$	-6
8_{13}	$S(29,11)$	$2t^{-2} - 7t^{-1} + 11 - 7t + 2t^2$	2
9_{14}	$S(37,14)$	$2t^{-2} - 9t^{-1} + 15 - 9t + 2t^2$	-2
9_{19}	$S(41,16)$	$2t^{-2} - 10t^{-1} + 17 - 10t + 2t^2$	-4
9_{27}	$S(49,18)$	$-t^{-3} + 5t^{-2} - 11t^{-1} + 15 - 11t + 5t^2 - t^3$	0

3. $SL(2,\mathbb{C})$ Casson invariant

We here recall briefly the definition of the $SL(2,\mathbb{C})$ Casson invariant, denoted by $\lambda_{SL(2,\mathbb{C})}$, based on [3]. Let M be a closed, orientable 3-manifold with a Heegaard splitting $H_1 \cup_F H_2$ with handlebodies H_1, H_2 and a Heegaard surface F, that is, $H_1 \cup H_2 = M$ and $\partial H_1 = \partial H_2 = H_1 \cap H_2 = F$. Then the inclusion maps $F \to H_i$ and $H_i \to M$ for $i = 1, 2$ induce surjections on the fundamental groups. It then follows that $X(M) = X(H_1) \cap X(H_2) \subset X(F)$, where $X(M)$, $X(H_1)$, $X(H_2)$ and $X(F)$ denote the $SL(2,\mathbb{C})$-character varieties for M, H_1, H_2 and F respectively. There are natural orientations on all the character varieties determined by their complex structures. The invariant $\lambda_{SL(2,\mathbb{C})}$ is (roughly) defined as an oriented intersection number of the subspaces of characters of irreducible representations in $X(H_1)$ and $X(H_2)$, which counts only compact, zero-dimensional components of the intersection. See [7] and [8], also [3] for detailed definition.

For the 3-manifolds obtained by Dehn surgeries on two-bridge knots, Boden and Curtis studied the $SL(2,\mathbb{C})$ Casson invariant $\lambda_{SL(2,\mathbb{C})}$ in detail in [3], and showed that $\lambda_{SL(2,\mathbb{C})}$ can be calculated as follows ([3, Theorem 2.5]): Let K be a two-bridge knot $S(\alpha,\beta)$ and $K(p/q)$ the 3-manifold obtained by p/q-surgery on K. Suppose that p/q is not a strict boundary slope and no p'-th root of unity is a root of $\Delta_K(t)$, where $p' = p$ if p is odd and $p' = p/2$ if p is even. Then

$$\lambda_{SL(2,\mathbb{C})}(K(p/q)) = \begin{cases} \dfrac{\|p/q\|_T}{2} & \text{if } p \text{ is even,} \\ \dfrac{\|p/q\|_T}{2} - \dfrac{\alpha-1}{4} & \text{if } p \text{ is odd.} \end{cases}$$

Here $\|p/q\|_T$ denotes the total Culler-Shalen seminorm of p/q.

Recall that a slope on the boundary of a knot exterior M is called a *boundary slope* if there exists an essential surface F embedded in M with nonempty boundary representing the slope, and a boundary slope is called *strict* if it is the boundary slope of an essential surface that is not the fiber of any fibration over the circle.

In this paper, we omit the detailed definition of the total Culler-Shalen seminorm (see [3] for example), while the calculation of the total Culler-Shalen seminorm of a slope for a two-bridge knot was essentially given in [22]. In fact, the following explicit formula is presented as [3, Proposition 2.3].

$$\|p/q\|_T = \frac{1}{2}\left(-|p| + \sum_i W_i \Delta(p/q, N_i)\right).$$

Here $N_1, \dots, N_n$ denote the boundary slopes for a two-bridge knot K, and $\Delta(p/q, N_i) := |p - qN_i|$ denotes the distance between slopes p/q and N_i, that is, the minimal geometric intersection number of the representatives of p/q and N_i. By the result given in [11], a boundary slope for a two-bridge knot $S(\alpha, \beta)$ is associated to a continued fraction expansion $[c, n_1, \dots, n_k]$ of β/α. Then W_i is set to be $\prod_j(|n_j| - 1)$ for the continued fraction expansion $[c, n_1, \dots, n_k]$ associated to N_i.

Combining these formulae, we see the following.

$$\begin{aligned}\lambda_{SL(2,\mathbf{C})}(K(p/q)) - \lambda_{SL(2,\mathbf{C})}(K(-p/q)) &= \frac{1}{2}\left(\left\|\frac{p}{q}\right\|_T - \left\|-\frac{p}{q}\right\|_T\right) \\ &= \frac{1}{4}\sum_i W_i\left(\Delta\left(\frac{p}{q}, N_i\right) - \Delta\left(-\frac{p}{q}, N_i\right)\right) \\ &= \frac{1}{4}\sum_i W_i(|p - qN_i| - |-p - qN_i|).\end{aligned}$$

In particular, we have the following when $p = 1$.

$$\begin{aligned}&\lambda_{SL(2,\mathbf{C})}(K(1/q)) - \lambda_{SL(2,\mathbf{C})}(K(-1/q)) \\ &\quad = \frac{1}{4}\sum_i W_i(|1 - qN_i| - |-1 - qN_i|) \\ &\quad = \frac{1}{4}\left(\sum_{N_i>0} W_i((qN_i - 1) - (1 + qN_i)) + \sum_{N_i<0} W_i((1 - qN_i) - (-1 - qN_i))\right) \\ &\quad = \frac{1}{2}\left(-\sum_{N_i>0} W_i + \sum_{N_i<0} W_i\right).\end{aligned}$$

Consequently, together with the result of Ni and Wu given in [19], a two-bridge knot has no purely cosmetic surgery pairs yielding homology 3-spheres if $-\sum_{N_i>0} W_i + \sum_{N_i<0} W_i \neq 0$ holds.

On the other hand, in [18, Theorem 2], the following method to enumerate all the continued fractions associated to boundary slopes for a two-bridge knot was given. The boundary slopes of a two-bridge knot $S(\alpha, \beta)$ are associated to the continued fractions obtained by applying the following substitutions at non-adjacent positions in the *simple continued fraction* (i.e., the unique one with all terms positive and the last term greater than 1) of β/α. The following exhibit the substitutions at position 2.

Substitution 1:

$$[c, n_1, 2n_2', n_3, n_4, \dots, n_k] \mapsto [c, n_1 + 1, (-2, 2)^{n_2' - 1}, -2, n_3 + 1, n_4, \dots, n_k]$$

Substitution 2:

$$[c, n_1, 2n_2' + 1, n_3, n_4, \dots, n_k] \mapsto [c, n_1 + 1, (-2, 2)^{n_2'}, -n_3 - 1, -n_4, \dots, -n_k]$$

Here, the notation $(-2,2)^n$ means that the pattern "$-2, 2$" is repeated n times.

Let us recall how to calculate the boundary slopes from a continued fraction.

By the result given in [11], a continued fraction expansion is associated to a boundary slope if it has partial quotients which are all at least two in absolute value. We call such a continued fraction a *boundary slope continued fraction*.

Given a two-bridge knot $S(\alpha, \beta)$, consider a boundary slope continued fraction expansion $[c, n_1, \dots, n_k]$ of β/α with integer part c and $|n_i| \geq 2$ for $1 \leq i \leq k$. Compare the signs of the terms $n_1, \dots, n_k$ to the pattern $[+ - + - \cdots]$, and let n^+ (resp. n^-) be the number of terms matching (resp. not matching) the pattern. Note that, among the boundary slope continued fractions, there is a unique one having all terms even; that is associated to the longitude (i.e., the boundary slope of a Seifert surface). Let n_0^+ and n_0^- be the corresponding values for the continued fraction associated to the longitude. Then, due to [11], the boundary slope associated to the continued fraction is presented as $2((n^+ - n^-) - (n_0^+ - n_0^-))$.

4. Calculation

In this section, we give a proof of Theorem 1.

As explained in the previous section, to prove the theorem, it suffices to enumerate all the boundary slopes by using the substitution method, and calculate $\sum_{N_i>0} W_i$ and $\sum_{N_i<0} W_i$ for the obtained boundary slopes.

First we consider the case $x = 1$, that is, the case of 9_{27}. We start with the simple continued fraction of $18/49$, which is represented as the continued fraction $[0, 2, 1, 2, 1, 1, 2]$. We use 6-tuples of the form $(b_1, b_2, b_3, b_4, b_5, b_6)$ with $b_j = 0, 1$ to show where substitutions are applied. As an example, $(0, 0, 1, 0, 0, 1)$ means the substitution rule is applied at positions 3 and 6. Then we have a boundary slope continued fraction $[0, 2, 2, -2, 2, 2, -2]$ which is the longitude continued fraction. Hence we see that $n_0^+ = 3$ and $n_0^- = 3$.

Here recall that each term of boundary slope continued fractions must be at least two in absolute value. Hence $(0, 0, 0, b_4, b_5, b_6)$ does not fit in our case since the term of 1 at position 2 remains after substitutions. Similarly, we can eliminate the possibility of $(b_1, b_2, 0, 0, 0, b_6)$ and $(b_1, b_2, b_3, 0, 0, 0)$. We also note that no two terms of 1 are adjacent in a 6-tuple. It is therefore enough to consider the following 10 cases to obtain all the boundary slope continued fractions.

Case 1. $(0,0,1,0,0,1)$.
Then we have $[0,2,2,-2,2,2,-2]$.
Hence $n_1^+ = 3$, $n_1^- = 3$ and $N_1 = 0$.

Case 2. $(0,0,1,0,1,0)$.
Then we have $[0,2,2,-2,3,-3]$.
Hence $n_2^+ = 1$, $n_2^- = 4$, $N_2 = -6$ and $W_2 = 4$.

Case 3. $(0,1,0,0,1,0)$.
Then we have $[0,3,-3,-2,3]$.
Hence $n_3^+ = 2$, $n_3^- = 2$ and $N_3 = 0$.

Case 4. $(0,1,0,1,0,0)$.
Then we have $[0,3,-4,2,2]$.
Hence $n_4^+ = 3$, $n_4^- = 1$, $N_4 = 4$ and $W_4 = 6$.

Case 5. $(0,1,0,1,0,1)$.
Then we have $[0,3,-4,3,-2]$.
Hence $n_5^+ = 4$, $n_5^- = 0$, $N_5 = 8$ and $W_5 = 12$.

Case 6. $(1,0,0,0,1,0)$.
Then we have $[1,-2,2,2,2,-3]$.
Hence $n_6^+ = 1$, $n_6^- = 4$, $N_6 = -6$ and $W_6 = 2$.

Case 7. $(1,0,0,1,0,0)$.
Then we have $[1,-2,2,3,-2,-2]$.
Hence $n_7^+ = 2$, $n_7^- = 3$, $N_7 = -2$ and $W_7 = 2$.

Case 8. $(1,0,0,1,0,1)$.
Then we have $[1,-2,2,3,-3,2]$.
Hence $n_8^+ = 3$, $n_8^- = 2$, $N_8 = 2$ and $W_8 = 4$.

Case 9. $(1,0,1,0,0,1)$.
Then we have $[1,-2,3,-2,2,2,-2]$.
Hence $n_9^+ = 2$, $n_9^- = 4$, $N_9 = -4$ and $W_9 = 2$.

Case 10. $(1,0,1,0,1,0)$.
Then we have $[1,-2,3,-2,3,-3]$.
Hence $n_{10}^+ = 0$, $n_{10}^- = 5$, $N_{10} = -10$ and $W_{10} = 8$.

We therefore see that

$$\lambda_{SL(2,\mathbf{C})}(M_{1/q}) - \lambda_{SL(2,\mathbf{C})}(M_{-1/q}) = \frac{1}{2}\left(-\sum_{N_i>0} W_i + \sum_{N_i<0} W_i\right) = -2.$$

Next we consider the general case, where $x \geq 2$.

We remark that the knot K_x is described as $K((8x^2-1)^2, 32x^3-8x^2-8x+2)$. Thus its simple continued fraction is given as $[0, 2x, 1, 1, 2x-2, 1, 2x-1, 1, 1, 2x-1]$. We in turn use 9-tuples of the form $(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9)$ with $b_j = 0, 1$ to show where substitutions are applied. The longitude continued fraction is obtained from $(0, 0, 1, 0, 1, 0, 0, 1, 0)$ and is $[0, 2x, 2, -2x, 2x, 2, -2x]$.

There is no possibility of $(0, 0, 0, b_4, b_5, b_6, b_7, b_8, b_9)$, $(b_1, 0, 0, 0, b_5, b_6, b_7, b_8, b_9)$, $(b_1, b_2, b_3, 0, 0, 0, b_7, b_8, b_9)$, $(b_1, b_2, b_3, b_4, b_5, 0, 0, 0, b_9)$ and $(b_1, b_2, b_3, b_4, b_5, b_6, 0, 0, 0)$ since each term of boundary slope continued fractions is at least two in absolute value. We again note that no two terms of 1 are adjacent in a 9-tuple. It is therefore enough to consider the following 25 cases to obtain all the boundary slope continued fractions.

Case 1. $(0, 0, 1, 0, 0, 1, 0, 0, 1)$.
Then we have $[0, 2x, 2, -2x+1, -2, (2, -2)^{x-1}, 2, 2, (-2, 2)^{x-1}]$.
Hence $n_1^+ = 2x+1$, $n_1^- = 2x+1$ and $N_1 = 0$.

Case 2. $(0, 0, 1, 0, 0, 1, 0, 1, 0)$.
Then we have $[0, 2x, 2, -2x+1, -2, (2, -2)^{x-1}, 3, -2x]$.
Hence $n_2^+ = 2x+2$, $n_2^- = 2$, $N_2 = 4x$ and $W_2 = 4(x-1)(2x-1)^2$.

Case 3. $(0, 0, 1, 0, 1, 0, 0, 1, 0)$.
Then we have $[0, 2x, 2, -2x, 2x, 2, -2x]$.
Hence $n_3^+ = 3$, $n_3^- = 3$ and $N_3 = 0$.

Case 4. $(0, 0, 1, 0, 1, 0, 1, 0, 0)$.
Then we have $[0, 2x, 2, -2x, 2x+1, -2, -2x+1]$.
Hence $n_4^+ = 2$, $n_4^- = 4$, $N_4 = -4$ and $W_4 = 4x(x-1)(2x-1)^2$.

Case 5. $(0, 0, 1, 0, 1, 0, 1, 0, 1)$.
Then we have $[0, 2x, 2, -2x, 2x+1, -3, (2, -2)^{x-1}]$.
Hence $n_5^+ = 1$, $n_5^- = 2x+2$, $N_5 = -4x-2$ and $W_5 = 4x(2x-1)^2$.

Case 6. $(0, 1, 0, 0, 0, 1, 0, 0, 1)$.
Then we have $[0, 2x+1, -2, -2x+2, -2, (2, -2)^{x-1}, 2, 2, (-2, 2)^{x-1}]$.
Hence $n_6^+ = 2x+2$, $n_6^- = 2x$, $N_6 = 4$ and $W_6 = 2x(2x-3)$.

Case 7. $(0, 1, 0, 0, 0, 1, 0, 1, 0)$.
Then we have $[0, 2x+1, -2, -2x+2, -2, (2, -2)^{x-1}, 3, -2x]$.
Hence $n_7^+ = 2x+3$, $n_7^- = 1$, $N_7 = 4x+4$ and $W_7 = 4x(2x-1)(2x-3)$.

Case 8. $(0, 1, 0, 0, 1, 0, 0, 1, 0)$.
Then we have $[0, 2x+1, -2, -2x+1, 2x, 2, -2x]$.
Hence $n_8^+ = 4$, $n_8^- = 2$, $N_8 = 4$ and $W_8 = 4x(x-1)(2x-1)^2$.

Case 9. $(0,1,0,0,1,0,1,0,0)$.
Then we have $[0, 2x+1, -2, -2x+1, 2x+1, -2, -2x+1]$.
Hence $n_9^+ = 3$, $n_9^- = 3$ and $N_9 = 0$.

Case 10. $(0,1,0,0,1,0,1,0,1)$.
Then we have $[0, 2x+1, -2, -2x+1, 2x+1, -3, (2,-2)^{x-1}]$.
Hence $n_{10}^+ = 2$, $n_{10}^- = 2x+1$, $N_{10} = -4x+2$ and $W_{10} = 16x^2(x-1)$.

Case 11. $(0,1,0,1,0,0,0,1,0)$.
Then we have $[0, 2x+1, -3, (2,-2)^{x-1}, -2x+1, -2, 2x]$.
Hence $n_{11}^+ = 2x+2$, $n_{11}^- = 1$, $N_{11} = 4x+2$ and $W_{11} = 8x(x-1)(2x-1)$.

Case 12. $(0,1,0,1,0,0,1,0,0)$.
Then we have $[0, 2x+1, -3, (2,-2)^{x-1}, -2x, 2, 2x-1]$.
Hence $n_{12}^+ = 2x+1$, $n_{12}^- = 2$, $N_{12} = 4x-2$ and $W_{12} = 8x(x-1)(2x-1)$.

Case 13. $(0,1,0,1,0,0,1,0,1)$.
Then we have $[0, 2x+1, -3, (2,-2)^{x-1}, -2x, 3, (-2,2)^{x-1}]$.
Hence $n_{13}^+ = 2x$, $n_{13}^- = 2x$ and $N_{13} = 0$.

Case 14. $(0,1,0,1,0,1,0,0,1)$.
Then we have $[0, 2x+1, -3, (2,-2)^{x-2}, 2, -3, (2,-2)^{x-1}, 2, 2, (-2,2)^{x-1}]$.
Hence $n_{14}^+ = 4x-1$, $n_{14}^- = 2x-1$, $N_{14} = 4x$ and $W_{14} = 8x$.

Case 15. $(0,1,0,1,0,1,0,1,0)$.
Then we have $[0, 2x+1, -3, (2,-2)^{x-2}, 2, -3, (2,-2)^{x-1}, 3, -2x]$.
Hence $n_{15}^+ = 4x$, $n_{15}^- = 0$, $N_{15} = 8x$ and $W_{15} = 16x(2x-1)$.

Case 16. $(1,0,0,1,0,0,0,1,0)$.
Then we have $[1, (-2,2)^x, 2, (-2,2)^{x-1}, 2x-1, 2, -2x]$.
Hence $n_{16}^+ = 2x+1$, $n_{16}^- = 2x+1$ and $N_{16} = 0$.

Case 17. $(1,0,0,1,0,0,1,0,0)$.
Then we have $[1, (-2,2)^x, 2, (-2,2)^{x-1}, 2x, -2, -2x+1]$.
Hence $n_{17}^+ = 2x$, $n_{17}^- = 2x+2$, $N_{17} = -4$ and $W_{17} = 2(x-1)(2x-1)$.

Case 18. $(1,0,0,1,0,0,1,0,1)$.
Then we have $[1, (-2,2)^x, 2, (-2,2)^{x-1}, 2x, -3, (2,-2)^{x-1}]$.
Hence $n_{18}^+ = 2x-1$, $n_{18}^- = 4x$, $N_{18} = -4x-2$ and $W_{18} = 2(2x-1)$.

Case 19. $(1,0,0,1,0,1,0,0,1)$.
Then we have $[1, (-2,2)^x, 2, (-2,2)^{x-2}, -2, 3, (-2,2)^{x-1}, -2, -2, (2,-2)^{x-1}]$.
Hence $n_{19}^+ = 4x-2$, $n_{19}^- = 4x-1$, $N_{19} = -2$ and $W_{19} = 2$.

Case 20. $(1,0,0,1,0,1,0,1,0)$.
Then we have $[1,(-2,2)^x,2,(-2,2)^{x-2},-2,3,(-2,2)^{x-1},-3,2x]$.
Hence $n_{20}^+ = 4x-1$, $n_{20}^- = 2x$, $N_{20} = 4x-2$ and $W_{20} = 4(2x-1)$.

Case 21. $(1,0,1,0,0,1,0,0,1)$.
Then we have $[1,(-2,2)^{x-1},-2,3,-2x+1,-2,(2,-2)^{x-1},2,2,(-2,2)^{x-1}]$.
Hence $n_{21}^+ = 2x$, $n_{21}^- = 4x$, $N_{21} = -4x$ and $W_{21} = 4(x-1)$.

Case 22. $(1,0,1,0,0,1,0,1,0)$.
Then we have $[1,(-2,2)^{x-1},-2,3,-2x+1,-2,(2,-2)^{x-1},3,-2x]$.
Hence $n_{22}^+ = 2x+1$, $n_{22}^- = 2x+1$ and $N_{22} = 0$.

Case 23. $(1,0,1,0,1,0,0,1,0)$.
Then we have $[1,(-2,2)^{x-1},-2,3,-2x,2x,2,-2x]$.
Hence $n_{23}^+ = 2$, $n_{23}^- = 2x+2$, $N_{23} = -4x$ and $W_{23} = 2(2x-1)^3$.

Case 24. $(1,0,1,0,1,0,1,0,0)$.
Then we have $[1,(-2,2)^{x-1},-2,3,-2x,2x+1,-2,-2x+1]$.
Hence $n_{24}^+ = 1$, $n_{24}^- = 2x+3$, $N_{24} = -4x-4$ and $W_{24} = 8x(x-1)(2x-1)$.

Case 25. $(1,0,1,0,1,0,1,0,1)$.
Then we have $[1,(-2,2)^{x-1},-2,3,-2x,2x+1,-3,(2,-2)^{x-1}]$.
Hence $n_{25}^+ = 0$, $n_{25}^- = 4x+1$, $N_{25} = -8x-2$ and $W_{25} = 8x(2x-1)$.

Since we are assuming $x \geq 2$,

$$\begin{aligned}
\lambda_{SL(2,\mathbf{C})}(M_{1/q}) - \lambda_{SL(2,\mathbf{C})}(M_{-1/q}) &= \frac{1}{2}\left(-\sum_{N_i>0} W_i + \sum_{N_i<0} W_i\right) \\
&= 8x^2 - 12x + 2 \\
&= 8\left(x - \frac{3}{4}\right)^2 - \frac{5}{2} \\
&> 0.
\end{aligned}$$

Remark 4. As stated in Introduction, our method can be applied further, but not sufficient even for knots with 10 crossings. In fact, there are two 2-bridge knots, 10_{33} and 10_{42}, which are unknown to admit purely cosmetic surgeries by using the (original) Casson invariant and the τ-invariant. Since the knot 10_{33} is amphicheiral, it has a symmetric boundary slope set, and so, our technique above is not applicable. Also the boundary slope set for the knot 10_{42} is symmetric, though the knot is not amphicheiral. Again our technique above is not applicable to this knot.

5. Alexander polynomial

In this section, we justify Remark 2 in Section 1 as follows.

PROPOSITION 2. *Let K_x be a two-bridge knot associated to the continued fraction expansion $[0, 2x, 2, -2x, 2x, 2, -2x]$ for a positive integer x. Then $\Delta''_{K_x}(1) = 0$ and $\tau(K_x) = 0$ hold. Here $\Delta_{K_x}(t)$ denotes the Alexander polynomial of K_x normalized to be symmetric and to satisfy $\Delta_{K_x}(1) = 1$.*

PROOF. Let K_x be a two-bridge knot associated to the continued fraction expansion $[0, 2x, 2, -2x, 2x, 2, -2x]$ for a positive integer x. Then K_x is a slice knot, originally observed by Casson and Gordon, and see [15, Lemma 8.2] for a proof. On the other hand, the invariant τ must vanish for slice knots as shown in [20, Corollary 1.3]. Thus we have $\tau(K_x) = 0$.

Now let us calculate the Alexander polynomial for K_x. This is just a straightforward calculation, but we include it for readers' convenience.

In general, a two-bridge knot associated to the continued fraction expansion $[0, 2A, -2B, 2C, -2D, 2E, -2F]$ is depicted as in Figure 2. Note that such a knot is of genus three, and any two-bridge knot of genus three admits such a description. In the figure, A to F denote the numbers of horizontal full-twists with signs of the twists.

Such a Seifert surface of genus three can be deformed into the one as shown in Figure 3. To calculate the Seifert matrix, we set a basis $a_1, \dots, a_6$ of the first homology group of the surface, as illustrated in Figure 3.

Then we have the Seifert matrix as follows.

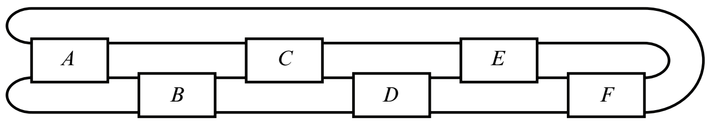

Fig. 2. A to F denote the numbers of full-twists.

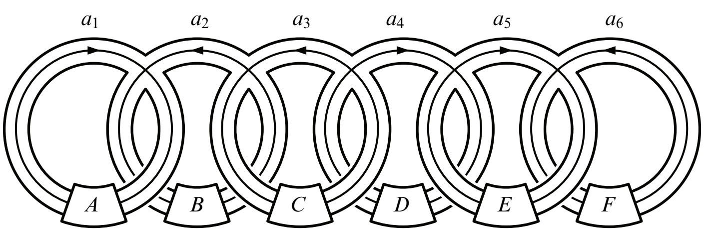

Fig. 3

$$M = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 1 & B & 1 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 1 & D & 1 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 1 & F \end{pmatrix}$$

Then $\Delta_{K_x}(t) = \det(M - t\,{}^tM)$ is obtained as

$$\det \begin{pmatrix} (1-t)A & -t & 0 & 0 & 0 & 0 \\ 1 & (1-t)B & 1 & 0 & 0 & 0 \\ 0 & -t & (1-t)C & -t & 0 & 0 \\ 0 & 0 & 1 & (1-t)D & 1 & 0 \\ 0 & 0 & 0 & -t & (1-t)E & -t \\ 0 & 0 & 0 & 0 & 1 & (1-t)F \end{pmatrix}$$

We then have the following polynomial of degree 6;

$$ABCDEF(1-t)^6 + ((A+C)DEF - ABC(D+F) + ABEF)t(1-t)^4$$
$$+ (AB+EF)t^2(1-t)^2 + t^3$$

Now we consider the form $[0, 2x, 2, -2x, 2x, 2, -2x]$, that is,

$$A = x, \qquad B = -1, \qquad C = -x, \qquad D = -x, \qquad E = 1, \qquad F = x.$$

This implies that $\Delta_{K_x}(t) = -x^4(1-t)^6 - x^2t(1-t)^4 + t^3$.
After normalization, we have the following.

$$\Delta_{K_x}(t) = -x^4(t^{-3} + t^3) + (6x^4 - x^2)(t^{-2} + t^2)$$
$$- (15x^4 - 4x^2)(t^{-1} + t) + 20x^4 - 6x^2 + 1$$

It follows that;

$$\Delta'_{K_x}(t) = -x^4(-3t^{-4} + 3t^2) + (6x^4 - x^2)(-2t^{-3} + 2t) + (-15x^4 + 4x^2)(-t^{-2} + 1)$$

$$\Delta''_{K_x}(t) = -x^4(12t^{-5} + 6t) + (6x^4 - x^2)(6t^{-4} + 2) + (-15x^4 + 4x^2)(2t^{-3})$$

$$\Delta''_{K_x}(1) = -18x^4 + 8(6x^4 - x^2) + 2(-15x^4 + 4x^2) = 0.$$

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During our study, computer-experiments were quite useful. The experiments were performed by using Dunfield's program [9], which implements Hatcher-Thurston's algorithm to enumerate boundary slopes for two-bridge knots.

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Appendix A. Another example

After writing up this paper, in [1], new criteria for knots in S^3 to admit purely cosmetic surgeries have been established. Those are given in terms of the Jones polynomial of knots, and actually, our examples K_x's in this paper can be shown to have no purely cosmetic surgeries by using them. However, the methods in that paper are completely different from this paper. Moreover, there exists an example of a knot which can not be identified to have no cosmetic surgeries by the criteria given in [1], but can be shown to admit no such surgeries yielding homology spheres by the method in this paper. In this appendix, we describe the example, that is, the knot $11a91$ in the knot table, depicted in Figure 4. See [1, Corollary 1.2].

We remark that the knot $11a91$ is the two-bridge knot $K(129,50)$ and the corresponding simple continued fraction is given as $[0,2,1,1,2,1,1,1,2]$. As in Section 4, we use 8-tuples of the form $(b_1,b_2,b_3,b_4,b_5,b_6,b_7,b_8)$ with $b_j=0,1$ to show where substitutions are applied. Then the longitude continued fraction is obtained from $(0,0,1,0,1,0,0,1)$ and is $[0,2,2,-4,2,2,-2]$.

For $11a91$, it is enough to consider the following 18 cases to obtain all the boundary slope continued fractions.

Case 1. $(0,0,1,0,0,1,0,0)$.
Then we have $[0,2,2,-3,-2,2,2]$.
Hence $n_1^+=3$, $n_1^-=3$ and $N_1=0$.

Case 2. $(0,0,1,0,0,1,0,1)$.
Then we have $[0,2,2,-3,-2,3,-2]$.
Hence $n_2^+=4$, $n_2^-=2$, $N_2=4$ and $W_2=4$.

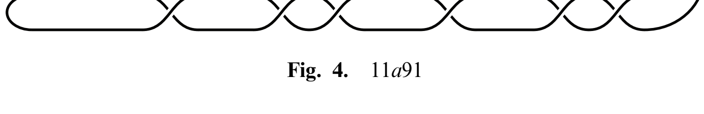

Fig. 4. $11a91$

Case 3. $(0,0,1,0,1,0,0,1)$.
Then we have $[0,2,2,-4,2,2,-2]$.
Hence $n_3^+ = 3$, $n_3^- = 3$ and $N_3 = 0$.

Case 4. $(0,0,1,0,1,0,1,0)$.
Then we have $[0,2,2,-4,3,-3]$.
Hence $n_4^+ = 1$, $n_4^- = 4$, $N_4 = -6$ and $W_4 = 12$.

Case 5. $(0,1,0,0,0,1,0,0)$.
Then we have $[0,3,-2,-2,-2,2,2]$.
Hence $n_5^+ = 4$, $n_5^- = 2$, $N_5 = 4$ and $W_5 = 2$.

Case 6. $(0,1,0,0,0,1,0,1)$.
Then we have $[0,3,-2,-2,-2,3,-2]$.
Hence $n_6^+ = 5$, $n_6^- = 1$, $N_6 = 8$ and $W_6 = 4$.

Case 7. $(0,1,0,0,1,0,0,1)$.
Then we have $[0,3,-2,-3,2,2,-2]$.
Hence $n_7^+ = 4$, $n_7^- = 2$, $N_7 = 4$ and $W_7 = 4$.

Case 8. $(0,1,0,0,1,0,1,0)$.
Then we have $[0,3,-2,-3,3,-3]$.
Hence $n_8^+ = 2$, $n_8^- = 3$, $N_8 = -2$ and $W_8 = 16$.

Case 9. $(0,1,0,1,0,0,1,0)$.
Then we have $[0,3,-3,2,-2,-2,3]$.
Hence $n_9^+ = 4$, $n_9^- = 2$, $N_9 = 4$ and $W_9 = 8$.

Case 10. $(0,1,0,1,0,1,0,0)$.
Then we have $[0,3,-3,2,-3,2,2]$.
Hence $n_{10}^+ = 5$, $n_{10}^- = 1$, $N_{10} = 8$ and $W_{10} = 8$.

Case 11. $(0,1,0,1,0,1,0,1)$.
Then we have $[0,3,-3,2,-3,3,-2]$.
Hence $n_{11}^+ = 6$, $n_{11}^- = 0$, $N_{11} = 12$ and $W_{11} = 16$.

Case 12. $(1,0,0,1,0,0,1,0)$.
Then we have $[1,-2,2,2,-2,2,2,-3]$.
Hence $n_{12}^+ = 3$, $n_{12}^- = 4$, $N_{12} = -2$ and $W_{12} = 2$.

Case 13. $(1,0,0,1,0,1,0,0)$.
Then we have $[1,-2,2,2,-2,3,-2,-2]$.
Hence $n_{13}^+ = 4$, $n_{13}^- = 3$, $N_{13} = 2$ and $W_{13} = 2$.

Case 14. $(1,0,0,1,0,1,0,1)$.
Then we have $[1,-2,2,2,-2,3,-3,2]$.
Hence $n_{14}^+ = 5$, $n_{14}^- = 2$, $N_{14} = 6$ and $W_{14} = 4$.

Case 15. $(1, 0, 1, 0, 0, 1, 0, 0)$.
Then we have $[1, -2, 3, -3, -2, 2, 2]$.
Hence $n_{15}^+ = 2$, $n_{15}^- = 4$, $N_{15} = -4$ and $W_{15} = 4$.

Case 16. $(1, 0, 1, 0, 0, 1, 0, 1)$.
Then we have $[1, -2, 3, -3, -2, 3, -2]$.
Hence $n_{16}^+ = 3$, $n_{16}^- = 3$ and $N_{16} = 0$.

Case 17. $(1, 0, 1, 0, 1, 0, 0, 1)$.
Then we have $[1, -2, 3, -4, 2, 2, -2]$.
Hence $n_{17}^+ = 2$, $n_{17}^- = 4$, $N_{17} = -4$ and $W_{17} = 6$.

Case 18. $(1, 0, 1, 0, 1, 0, 1, 0)$.
Then we have $[1, -2, 3, -4, 3, -3]$.
Hence $n_{18}^+ = 0$, $n_{18}^- = 5$, $N_{18} = -10$ and $W_{18} = 24$.

Therefore,

$$\begin{aligned}\lambda_{SL(2,\mathbb{C})}(M_{1/q}) - \lambda_{SL(2,\mathbb{C})}(M_{-1/q}) &= \frac{1}{2}\left(-\sum_{N_i>0} W_i + \sum_{N_i<0} W_i\right) \\ &= 6 \\ &> 0.\end{aligned}$$

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