

Connection Formula of Basic Hypergeometric Series ${}_r\phi_{r-1}(\mathbf{0}; \mathbf{b}; q, x)$

By

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Abstract

We show a connection formula of a linear q -differential equation satisfied by ${}_r\phi_{r-1}(\mathbf{0}; \mathbf{b}; q, x)$ where any element of \mathbf{b} are not zero. We use a q -Laplace transformation to obtain an integral representation of solutions of the q -differential equation.

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1 Introduction

We show a connection formula of a linear q -differential equation satisfied by ${}_r\phi_{r-1}(0, 0, \dots, 0; b_1, \dots, b_{r-1}; q, x)$ in case that $b_1 b_2 \cdots b_{r-1} \neq 0$. The basic hypergeometric series ${}_r\phi_{r-1}(0, 0, \dots, 0; b_1, \dots, b_{r-1}; q, x)$ satisfies a linear q -differential equation of the r -th order:

$$\left[x - (1 - \sigma_q) \prod_{k=1}^{r-1} \left(1 - \frac{b_k}{q} \sigma_q \right) \right] y(x) = 0, \quad (1)$$

where $\sigma_q y(x) = y(xq)$. The condition $b_1 b_2 \cdots b_{r-1} \neq 0$ implies that the origin is a regular singular point of (1). Around the infinity (1) has r solutions which are represented by convergent power series on $x^{1/r}$. In this sense, (1) is the most degenerate case of hypergeometric equations.

Thomae [6, 7] showed a connection formula on ${}_2\phi_1(a_1, a_2; b_1)$ and ${}_3\phi_2(a_1, a_2, a_3; b_1, b_2)$. In [8] Watson gave connection formulae in more general cases. He showed a connection formula of ${}_r\phi_{r-1}(a_1, a_2, \dots, a_r; b_1, \dots, b_s, \mathbf{0}; q, z)$, where $s < r$. Watson also showed that an asymptotic expansion of ${}_{s+1}\phi_{r-1}(a'_1, a'_2, \dots, a'_{s+1}; b'_1, \dots, b'_{r-1}; q, z)$ (he used a notation ${}_{s+1}\mathfrak{P}_{r-1}$), but he did not give a resummation of divergent series. Later Slater [4, 5] also gave a more general form of a connection formula.

J.-P. Ramis, J. Sauloy and C. Zhang started modern study on divergent q -series and a q -analogue of the Stokes phenomenon [3]. Zhang studied the q -Stokes phenomenon of q -confluent hypergeometric function ${}_2\phi_0(a, b; 0; q, x)$ [10]. He has also shown a connection formula of Jackson's q -analogue of the Bessel function $J_\nu^{(1)}$ [11]. Since

$$J_\nu^{(1)}(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \left(\frac{x}{2}\right)^\nu {}_2\phi_1\left(0, 0; q^{\nu+1}; q, -\frac{x^2}{4}\right),$$

the connection formula of $J_\nu^{(1)}(x; q)$ is essentially the case $r = 2$ of (1).

Since all of local solutions around the origin and the infinity are represented by convergent power series, we can determine the connection formula by a q -Laplace transformation [9]. We show a useful formula on p -Laplace transformation applied to q -difference equations ($p^m = q$) in section two.

We show a connection formula in section three. We study the q -differential equation

$$\left[x^r \prod_{k=1}^r (1 - a_k \sigma_p) - \left(-\frac{\sigma_p}{p^r} \right)^r \right] u(x) = 0.$$

Local solutions around the infinity are

$$u_{1,\infty}(x) = \frac{\theta_p(-a_1 x)}{\theta_p(-x)} {}_r\phi_{r-1}\left(0, 0, \dots, 0; p^r a_1/a_2, p^r a_2/a_3, \dots, p^r a_1/a_r; p^r, \frac{1}{a_1 a_2 \cdots a_r x^r}\right)$$

and $u_{2,\infty}(x), \dots, u_{r,\infty}(x)$ are obtained by the cyclic transformation of a_1, a_2, \dots, a_r .

We take a primitive r -th root ω of unity. Local solutions around the origin are

$$u_{j,0}(x) = \frac{1}{\theta_p(-\omega^j p^{(1-r)/2} x)} v_j(x), \quad v_j(x) = \sum_{n=0}^{\infty} v_n^{(j)} x^n,$$

for $j = 0, 1, 2, \dots, r-1$. We assume that $v_0^{(j)} = 1$. The connection formula between $(u_{0,0}, u_{1,0}, \dots, u_{r-1,0})$ and $(u_{1,\infty}, \dots, u_{r,\infty})$ is given by

$$v_j(x) = \frac{1}{(q, a_2/a_1, \dots, a_r/a_1; q)_\infty} \frac{\theta_p(-\omega^j p^{(1-r)/2} a_1 x) \theta_p(-x)}{\theta_p(-\omega^j p^{(1-r)/2} x) \theta_p(-a_1 x)} u_{1,\infty}(x) \\ + \text{idem}(a_1; a_2, \dots, a_r).$$

The symbol "idem $(a_1; a_2, \dots, a_r)$ " stands for the sum of the r expressions obtained from the preceding expression by interchanging a_1 with each a_2, a_3, \dots, a_r .

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2 Notations

We denote the m -vetcor $(0, 0, \dots, 0)$ by $\mathbf{0}_m$.

We assume that $0 < |q| < 1$. For $n = 0, 1, 2, \dots$, we set the q -shifted factorial

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j), \quad (a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j).$$

We set $(a_1, a_2, \dots, a_m; q)_n = \prod_{j=1}^m (a_j; q)_n$ for $n = 0, 1, 2, \dots$ or $n = \infty$.

We set the theta function

$$\theta_q(x) := \theta(x) = \sum_{k \in \mathbb{Z}} q^{k(k-1)/2} x^k = (q, -x, -q/x; q)_\infty.$$

The theta function satisfies

$$\begin{aligned} \theta_q(q^k x) &= q^{-k(k-1)/2} x^{-k} \theta_q(x) \quad (k \in \mathbb{Z}), \\ x\theta_q(1/x) &= \theta_q(x), \quad \theta_q(1/x) = \theta_q(qx). \end{aligned}$$

The basic hypergeometric series [1] is defined by

$$\begin{aligned} &{}_r\phi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) \\ &:= \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \end{aligned}$$

Let σ_q be a q -shift operator $\sigma_q[f(x)] = f(xq)$. When $1 + s \geq r$, ${}_r\phi_s$ is convergent and satisfies a q -difference equation with $(s + 1)$ -th order

$$\left[x \prod_{j=1}^r (1 - a_j \sigma_q) - (1 - \sigma_q) \prod_{k=1}^{r-1} \left(1 - \frac{b_k}{q} \sigma_q \right) \right] y(x) = 0.$$

3 q -Borel transformation and q -Laplace transformation

We review a q -Borel transformation and a q -Laplace transformation. See [9, 3] for detail.

The q -Borel transformation $\mathcal{B}_q^- : \mathbb{C}[[x]] \rightarrow \mathbb{C}[[\tau]]$ is defined by

$$\mathcal{B}_q^- \left[\sum_{n=0}^{\infty} a_n x^n \right] := \sum_{n=0}^{\infty} a_n q^{-n(n-1)/2} \tau^n.$$

We identify a germ of holomorphic functions at the origin $\mathcal{O}_{\mathbb{C},0}$ as a subset of $\mathbb{C}[[x]]$. As a linear operator on $\mathbb{C}[[x]]$, the following lemma is useful to study q -difference equations.

Lemma 1. (1) The q -Borel transformation \mathcal{B}_q^- shifts the power of σ_q :

$$\mathcal{B}_q^-(x^m \sigma_q^n f) = q^{-m(m-1)/2} \tau^m \sigma_q^{n-m} \mathcal{B}_q^-(f).$$

(2) Multiplication by the theta function shifts the power of x :

$$x^m \sigma_q^n \left[\frac{1}{\theta_q(cx)} f(x) \right] = \frac{q^{n(n-1)/2} c^n}{\theta_q(cx)} x^{m+n} \sigma_q^n f(x).$$

The inverse transformation of \mathcal{B}_q^- is given by a q -Laplace transform \mathcal{L}_q^- . Assume that $\varphi(\tau)$ is holomorphic on $|\tau| \leq \varepsilon$. We define

$$\mathcal{L}_q^- \varphi(x) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \varphi(\tau) \theta_q(x/\tau) \frac{d\tau}{\tau}.$$

Under a suitable condition, we have $\mathcal{L}_q^- \circ \mathcal{B}_q^- f = f$.

We consider the p -Laplace transform of a ratio of p^m -products.

Proposition 2. Let m be a positive integers. We set $p^m = q$. We assume that $s + m \leq r$. When $s + m = r$, we need $|q^{(1+m)/2} b_1 \cdots b_s / a_1 a_2 \cdots a_r x^m| < 1$. We consider the contour integral

$$I = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \frac{\prod_{j=1}^s (b_j \tau; q)_{\infty}}{\prod_{k=1}^r (a_k \tau; q)_{\infty}} \theta_p(x/\tau) \frac{d\tau}{\tau},$$

where $\prod_{k=1}^r (a_k \tau; q)_{\infty}$ does not have any zero on $|\tau| \leq \varepsilon$. Then we obtain

$$\begin{aligned} I &= \frac{(b_1/a_1, \dots, b_s/a_1; q)_{\infty}}{(q, a_2/a_1, \dots, a_r/a_1; q)_{\infty}} \theta_p(a_1 x) \\ &\quad \times {}_{s+m} \phi_{r-1} \left(\begin{matrix} qa_1/b_1, \dots, qa_1/b_s, \mathbf{0}_m \\ qa_1/a_2, \dots, qa_1/a_r \end{matrix}; q, \frac{(-1)^r q^{r-s+(1-m)/2} b_1 \cdots b_s}{a_1^{m+s-r+1} a_2 \cdots a_r x^m} \right) \\ &\quad + \text{idem}(a_1; a_2, \dots, a_r). \end{aligned} \quad (2)$$

Proof. The following relations are directly proved :

$$\text{Res}_{\tau=1/aq^n} \frac{1}{(a\tau; q)_{\infty}} \frac{d\tau}{\tau} = -\frac{(-1)^n q^{n(n+1)/2}}{(q; q)_{\infty} (q; q)_n},$$

$$\begin{aligned}\theta_p(x/\tau)|_{\tau \rightarrow 1/aq^n} &= (ax)^{-mn} p^{-nm(nm-1)/2} \theta_p(ax), \\ (b\tau; q)_\infty|_{\tau \rightarrow 1/aq^n} &= (-b/a)^n q^{-n(n+1)/2} (b/a; q)_\infty (aq/b; q)_n.\end{aligned}$$

By using the above relations we can show Proposition. \square

4 Connection formula

We consider the equation

$$\left[z \prod_{j=k}^r (1 - a_k \sigma_q) - \left(-\frac{\sigma_q}{q} \right)^r \right] y(z) = 0. \quad (3)$$

Local solutions of (3) around the infinity are

$$y_{1,\infty}(z) = \frac{\theta_q(-a_1 z)}{\theta_q(-z)} {}_r\phi_{r-1} \left(\begin{matrix} 0, 0, \dots, 0 \\ qa_1/a_2, qa_j/a_3, \dots, qa_1/a_r \end{matrix}; q, \frac{1}{a_1 \cdots a_r z} \right)$$

and $y_{2,\infty}(z), \dots, y_{r,\infty}(z)$ are obtained by the cyclic transform $a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_r \rightarrow a_1$.

Since (3) has ramified solutions around the origin, we take a covering transformation $z = x^r$. We set $p^r = q$.

$$\left[x^r \prod_{k=1}^r (1 - a_k \sigma_p) - \left(-\frac{\sigma_p}{q} \right)^r \right] u(x) = 0. \quad (4)$$

We give a connection formula of (4). Local solutions of (4) around the infinity are

$$u_{1,\infty}(x) = \frac{\theta_p(-a_1 x)}{\theta_p(-x)} {}_r\phi_{r-1} \left(\begin{matrix} 0, 0, \dots, 0 \\ qa_1/a_2, qa_2/a_3, \dots, qa_1/a_r \end{matrix}; q, \frac{1}{a_1 a_2 \cdots a_r x^r} \right)$$

and $u_{2,\infty}(x), \dots, u_{r,\infty}(x)$ are obtained by the cyclic transform.

We take a complex number ω , which is a primitive r -th root of unity: $\omega^r = 1$. Local solutions of (4) around the origin are

$$u_{j,0}(x) = \frac{1}{\theta_p(-\omega^j p^{(1-r)/2} x)} v_j(x), \quad v_j(x) = \sum_{n=0}^{\infty} v_n^{(j)} x^n,$$

for $j = 0, 1, 2, \dots, r-1$. We assume that $v_0^{(j)} = 1$. We show a connection formula between $(u_{0,0}, u_{1,0}, \dots, u_{r-1,0})$ and $(u_{1,\infty}, \dots, u_{r,\infty})$.

We set elementary symmetric polynomials s_1, s_2, \dots, s_r so that

$$\prod_{k=1}^r (1 - a_k x) = \sum_{k=1}^r (-1)^k s_k x^k.$$

We set $c_j = -\omega^j p^{(1-r)/2}$. Then $v_j(x)$ satisfies a q -difference equation

$$\left[\sum_{k=1}^r (-1)^k c_j^k p^{k(k-1)/2} s_k x^k \sigma_p^k - \sigma_p^r \right] v_j(x) = 0.$$

We remark that $\sigma_p^r = \sigma_q$. Since $w_j(\tau) = (\mathcal{B}_p^- v_j)(\tau)$ satisfies

$$\left[\prod_{k=1}^r (1 - c_j a_k \tau) - \sigma_p^r \right] w_j(\tau) = 0,$$

we have

$$w_j(\tau) = \frac{1}{(c_j a_1 \tau, c_j a_2 \tau, \dots, c_j a_r \tau; q)_\infty}.$$

By (2) we obtain

$$\begin{aligned} v_j(x) &= \mathcal{L}_p^- w_j(x) = \frac{1}{2\pi i} \int_{|\tau|=\varepsilon} \frac{1}{(c_j a_1 \tau, c_j a_2 \tau, \dots, c_j a_r \tau; q)_\infty} \theta_p(x/\tau) \frac{d\tau}{\tau} \\ &= \frac{\theta_p(c_j a_1 x)}{(q, a_2/a_1, \dots, a_r/a_1; q)_\infty} {}_r\phi_{r-1} \left(\begin{matrix} 0, 0, \dots, 0 \\ qa_1/a_2, \dots, qa_1/a_r \end{matrix}; q, \frac{(-1)^r q^{(1-r)/2}}{c_j^r a_1 a_2 \cdots a_r x^r} \right) \\ &\quad + \text{idem}(a_1; a_2, \dots, a_r) \\ &= \frac{\theta_p(c_j a_1 x)}{(q, a_2/a_1, \dots, a_r/a_1; q)_\infty} {}_r\phi_{r-1} \left(\begin{matrix} 0, 0, \dots, 0 \\ qa_1/a_2, \dots, qa_1/a_r \end{matrix}; q, \frac{1}{a_1 a_2 \cdots a_r x^r} \right) \\ &\quad + \text{idem}(a_1; a_2, \dots, a_r). \end{aligned}$$

We remark that $c_j^r = (-1)^r q^{(1-r)/2}$.

The main result is as follows:

Theorem 3. *We take a primitive r -th root ω of unity. A connection formula of (4) is given by*

$$\begin{aligned} u_{j,0}(x) &= \frac{1}{(q, a_2/a_1, \dots, a_r/a_1; q)_\infty} \frac{\theta_p(-\omega^j p^{(1-r)/2} a_1 x) \theta_p(-x)}{\theta_p(-\omega^j p^{(1-r)/2} x) \theta_p(-a_1 x)} u_{1,\infty}(x) \\ &\quad + \text{idem}(a_1; a_2, \dots, a_r), \end{aligned}$$

for $j = 0, 1, \dots, r-1$.

The case $r = 2$:

We set $p^2 = q$. We take a p -difference equation

$$[p^2 x^2 (1 - a_1 \sigma_p)(1 - a_2 \sigma_p) - \sigma_p^2] u(x) = 0. \quad (5)$$

We give a connection formula of (5). Local solutions of (5) around the infinity are

$$\begin{aligned} u_{1,\infty}(x) &= \frac{\theta_p(-a_1x)}{\theta_p(-x)} {}_2\phi_1\left(0, 0; qa_1/a_2; q, \frac{1}{a_1a_2x^2}\right), \\ u_{2,\infty}(x) &= \frac{\theta_p(-a_2x)}{\theta_p(-x)} {}_2\phi_1\left(0, 0; qa_2/a_1; q, \frac{1}{a_1a_2x^2}\right). \end{aligned}$$

Local solutions of (5) around the origin are

$$\begin{aligned} u_{0,1}(x) &= \frac{1}{\theta_p(-p^{-1/2}x)} v_1(x), & v_1(x) &= \sum_{n=0}^{\infty} v_n^{(1)} x^n, \\ u_{0,2}(x) &= \frac{1}{\theta_p(p^{-1/2}x)} v_2(x), & v_2(x) &= \sum_{n=0}^{\infty} v_n^{(2)} x^n. \end{aligned}$$

We assume that $v_0^{(j)} = 1$ for $j = 1, 2$. By Theorem 3 we obtain

$$\begin{aligned} u_{0,1}(x) &= \frac{1}{(q, a_2/a_1; q)_{\infty}} \frac{\theta_p(-p^{1/2}a_1x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_1x)} u_{1,\infty}(x) \\ &\quad + \frac{1}{(q, a_1/a_2; q)_{\infty}} \frac{\theta_p(-p^{1/2}a_2x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_2x)} u_{2,\infty}(x), \\ u_{0,2}(x) &= \frac{1}{(q, a_2/a_1; q)_{\infty}} \frac{\theta_p(p^{1/2}a_1x)\theta_p(-x)}{\theta_p(p^{1/2}x)\theta_p(-a_1x)} u_{1,\infty}(x) \\ &\quad + \frac{1}{(q, a_1/a_2; q)_{\infty}} \frac{\theta_p(-p^{1/2}a_2x)\theta_p(-x)}{\theta_p(-p^{1/2}x)\theta_p(-a_2x)} u_{2,\infty}(x). \end{aligned}$$

This connection formula is essentially equivalent to the connection formula of Jackson's first q -Bessel functions in [11].

5 Conclusion

We show a connection formula of (4), which is a generalization of Jackson's first q -analogue of the Bessel functions [2]. We can obtain a connection formula of solutions represented by a convergent (non-hypergeometric) series of $x^{1/m}$ by applying the p -Laplace transformation (2) to a product of p^m -shifted factorials p for other q -hypergeometric equations.

We should study the q -Stokes phenomenon [3] for divergent series solutions. By using the other q -Borel transformation \mathcal{B}_q^+ , we can give a resummation for divergent hypergeometric series.

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